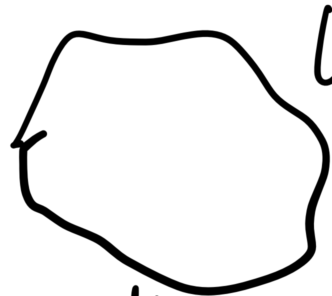
Lec2-3

Essential Theory of ODE

unknown $x: I \to \mathbb{R}$, $I \subset \mathbb{R}$

Main focus: nonlinear systems of first order ODE

Def $U \subset \mathbb{R}^n$ open set $f: U \to \mathbb{R}^n$



$U \subset \mathbb{R}^n$

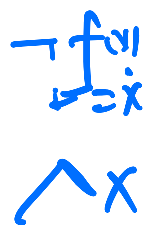
$x(t)$ is a solution of ODE

$\dot{x} = f(x)$. on an interval I

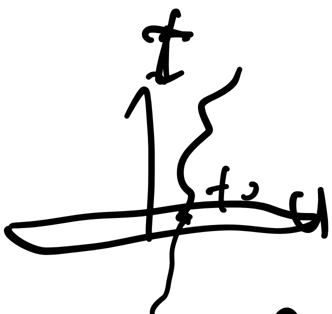
if for all $t \in I$, x is differentiable

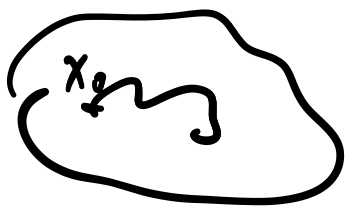
$x(t) \in U$

$\dot{x}(t) = f(x(t))$



ODE





Given $x_0 \in U$. $t_0 \in I$ if $\dot{x} = f(x)$

$x(t_0) = x_0$.

satis IVP

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

Remark. If f: continuous, then $\dot{x}$ continuous $\to x \in C^1$

$\boxed{\dot{x} = f(x)}$. If $f \in C^1$, then $\dot{x} \in C^1$ (chain rule) $\to x \in C^2$

If $f \in C^k \to x \in C^{k+1}$. bootstrapping.

First main goal. well-posedness.

Existence uniqueness. continuous.

Example: Lotka-Volterra model. (predator-prey model)

x_2 x_1 over time t

Given. $\alpha, \beta, \gamma, \delta$.

$$\begin{cases} \dot{x}_1 = \alpha x_1 - \beta x_1 x_2 \\ \dot{x}_2 = \delta x_1 x_2 - \gamma x_2 \end{cases}$$

if no predator. prey grows exponentially (α)

if no prey. predator decays exponentially ($-\gamma$)

predation rate $\propto$ number of encounters

$\propto$ the product of the population.

(assume uniform)

$f: \mathbb{R}^2 \to \mathbb{R}^2$

$f(x_1, x_2) = (\alpha x_1 - \beta x_1 x_2, \delta x_1 x_2 - \gamma x_2)$

well-poseness. $\exists$! give a prediction

Stability. will not change significantly

Why first-order? WLOG

Take a k^{th} order $x^{(k)} = g(x, x' \cdots x^{(k-1)})$

Let $y_1 = x$, $y_2 = x' \cdots y_k = x^{(k-1)}$

$$\begin{cases} \dot{y}_1 = \dot{x} = y_2 \\ \dot{y}_2 = x'' = y_3 \\ \vdots \\ \dot{y}_k = x^{(k)} = g(x, x' \cdots x^{(k-1)}) \end{cases}$$

k^{th} order ODE $\longrightarrow$ 1^{st} order $\dot{y} = F(y)$

$F(y) = (y_1, y_2, y_3 \cdots y_{k-1}, g(y, \cdots y_{k-1}))$.

Non-autonomous $\dot{x} = f(x, t)$

why autonomous ODE? WLOG

$x : \dot{x} = g(x, t)$

Let $y = (x, t)$

y solves $\dot{y} = f(y)$, $f(y) = f(x, t) = (g(y), 1)$

$\dot{y} = (\dot{x}, \frac{dt}{dt})$

Structural Assumptions on f

$\boxed{H_1}$: $f: U \to \mathbb{R}^n$ continuous, $U \subset \mathbb{R}^n$ open. $\leftrightarrow x \in C^1$

$\boxed{H_2}$. f is locally Lipschitz.

i.e. for each compact set $K \subset U$

$\exists\ L_K > 0$, s.t.

$|f(p) - f(q)| \leq L|p-q|, \forall p, q \in K$

Thm. Fundamental existence and uniqueness thm

(Picard / Cauchy Lip)

f: H_1, H_2, Fix $x_0 \in U$

$\exists\ a > 0$ s.t. the IVP

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

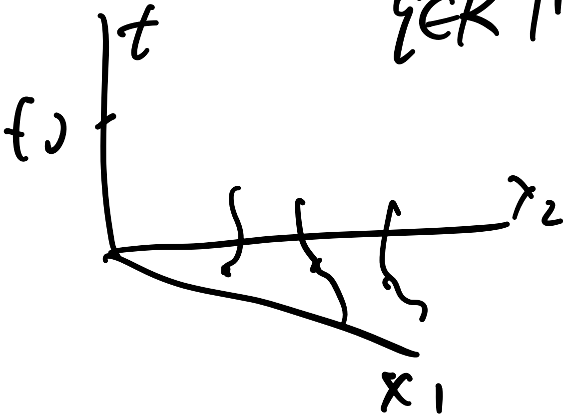
has a unique solution on interval $[t_0 - a, t_0 + a]$

More concretely. if $r > 0$ s.t. $K := \overline{B(x_0, r)} \subset U$

L Lipschitz for f on K

$M = \sup |f(q)|$ we can take

$\xi \in K$ $\|f'(\xi)\|$ any $a < \min\{\frac{1}{C}, \frac{r}{M}\}$



Proof.

x solves IVP $\Leftrightarrow$ x is continuous. satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))\,ds \quad (*)$$

$\Leftarrow$:

$x(t_0) = x_0$ ✓

f is continuous. x is differentiable at t. (fundamental ~~theorem~~ Calculus) (FTC).

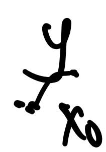
since $f \circ x$ contin (x is contin)

$x'(t) = f(x(t))$ ✓

So. we show $\exists$ $x(t) = x_0 + \int_{t_0}^{t} f(x(s))\,ds$ (*)

fixed point problem.

$y \in \underbrace{C^0}_{\text{contin.}}([t_0 - a, t_0 + a], U)$



def: $F(y)(t) = x_0 + \int_{t_0}^{t} f(y(s))\,ds$

x solves $(*)$ $\Longleftrightarrow$ x is a fixed point of F.

i.e. $F(x) = x$

Key tool. Banach fixed point theorem (contraction mapping)

Let (X,d) be nonempty complete metric spac

$C[I;\mathbb{R}]$

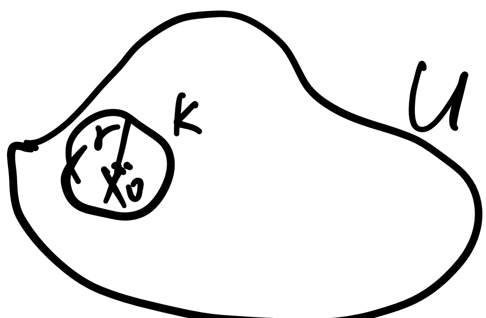
$F: X \to X$ is a contract mapping:

$\exists\, k \in [0,1)$ s.t.

$d(F(x), F(y)) \le k\, d(x,y) \quad \forall\, x, y \in X$

Then $\exists!$ fixed point x^* of F $(x^* \in X)$

$F(x^*) = x^*$



Since U open $\Rightarrow \exists\, r > 0$, s.t. $K := \overline{B(x_0, r)} \subset U$ $\triangle$

By H_2, $|f(x) - f(y)| \le L|x-y| \quad \forall\, x, y \in K$.

By H_1, continuous function is bounded on compact set.

$\Rightarrow \|f(x)\|_\infty \le M, \ \forall x \in K$

Fix $a < \min\{\frac{1}{L}, \frac{r}{M}\}$ let $I := [t_0 - a, t_0 + a]$

$X = C^0(I, K)$ equipped with $d(u,v) = \sup_{t \in I} |u(t) - v(t)|$

ex: Verify (X, d) is a complete metric space.

$(C^0, \|f\|_\infty)$ complete

Given $y \in X$. define.

$$F(y)(t) = x_0 + \int_{t_0}^{t} f(y(s))\,ds$$

Claim 1. $F: X \to X$

Proof of claim 1: Take $y \in C^0(I, K) = X$

FTC $\Rightarrow F(y) \in C^0(I, \mathbb{R}^n)$

NTS. this can (need to show) be replaced by $K \in \mathbb{R}^n$

To see $F(y)$ takes values in $K = \overline{B}(x_0, r)$

$$\left|F(y)(t) - x_0\right| = \left|\int_{t_0}^{t} f(y(s))\,ds\right|$$

Abs. ineq.

if $t \ge t_0$ $\le \int_{t_0}^{t} \ldots$

at least sup limit of integrate $\leq \int_{t_0}^{t} |f(y(s))| ds$

$\leq M|t-t_0|$ $\begin{cases} y \in K. \ \boxed{\|f(y)\|_\infty \leq M} \\ y(s) \in K, \ \forall s \in [t_0-a, t_0+a] \end{cases}$

$\leq Ma < r$

$(a < \frac{M}{r})$

$\Rightarrow F(y)(t) \in B(x_0, r) \subset K$

Claim 2. $F: X \to X$ is a contraction mapping.

Take $x_1, x_2 \in X$

$d(F(x_1), F(x_2)) = \sup_{t \in I} |F(x_1)(t) - F(x_2)(t)|$

$= \sup_{t \in I} \left| \int_{t_0}^{t} |f(x_1(s)) - f(x_2(s))| ds \right|$

abs. tri-eq $\leq \int_{t}^{t_0} | \ | ds$

$\leq \int_{t_0-a}^{t_0} | \ | ds$

WLOG sup achieved in $[t_0-a, t_0]$ $\leq \int_{t_0-a}^{t_0} |f(x_1(s)) - f(x_2(s))| ds$

$\leq L \int_{t_0-a}^{t_0} |x_1(s) - x_2(s)| ds$

$\leq a \cdot \max_{s \in [t_0-a, t_0]} |x_1(s) - x_2(s)|$

$\leq \underbrace{\boxed{L \cdot a}}_{:= K < 1} d(x_1, x_2)$

By Banach fix. point thm.

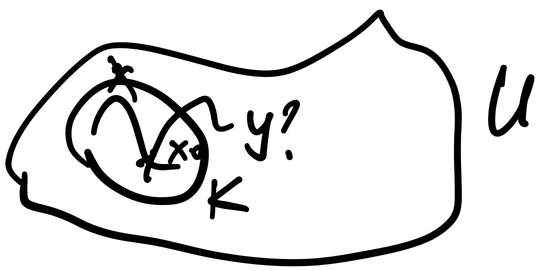
$\Rightarrow f$ has unique fixed point

$x \in X = C^0(I, K)$

$\Rightarrow x$ solves (IVP) on I

$\Rightarrow$ existence & uniqueness.

only solution to IVP.



that f completely takes values in $\boxed{\text{in } K}$

Full uniqueness. Gronwall-type argument.

Let y_1, y_2 be two solns to IVP on $\boxed{I}$

$\tilde{K} := y_1(I) \cup y_2(I)$ image

compact $\exists\, C > 0$.

Lipschitz valid on $\tilde{K}$

(H2)

Same I for y_1, y_2.

$$\underbrace{|y_1(t) - y_2(t)|}_{H'(t)} \leq \left| \int_{t_0}^{t} f(y_1(s)) - f(y_2(s))\, ds \right| \overset{\text{triang. l.}}{\leq} \int_{t_0}^{t} |f(y_1(s)) - f(y_2(s))|\, ds \leq \tilde{L} \underbrace{\int_{t_0}^{t} |y_1(s) - y_2(s)|\, ds}_{=:H(t)}$$

$H'(t) - \tilde{L} H(t) \leq 0$ differential inequality

Multi by $e^{-\tilde{L} t} \geq 0$

$$0 \geq e^{-\tilde{L} t} \left(H'(t) - \tilde{L} H(t) \right) = \left[e^{-\tilde{L} t} H(t) \right]'$$

$\Rightarrow e^{-\tilde{L} t} H(t)$ non-increasing

$t \in [t_0, t_0 + a]$

$$0 = 0 = e^{-\tilde{L} t_0} \underbrace{H(t_0)}_{\int_{t_0}^{t_0} = 0} \geq \underbrace{e^{-\tilde{L} t}}_{\geq 0} H(t) \geq 0$$

$$\Downarrow$$

$H(t) \equiv 0$

$\Rightarrow y_1 = y_2$ on I

Gronwall's Lemma (Integral form)

Suppose $f: [t_0, t_1] \to [0, +\infty)$ continuous.

$f(t) \le C + \int_{t_0}^{t} k f(s) ds$, $\forall t \in [t_0, t_1]$

$C \in \mathbb{R}$, $k \ge 0$,

$$f(t) \le C e^{k(t-t_0)} \quad \forall t \in [t_0, t_1]$$

Proof. Let $G(t) := C + \int_{t_0}^{t} k f(s) ds$, $G \in C^1$

$f(t) \le G(t)$ so that $G'(t) = k f(t) \le k G(t)$
$(k \ge 0)$

$\underline{G'(t) - k G(t) \le 0}$

Multiply. e^{-kt}

$$(G(t) e^{-kt})' \le 0$$

$$G(t_0) e^{-k t_0} \ge G(t) e^{-kt} \quad \forall t \in [t_0, t_1]$$

$$f(t) \le G(t) \le e^{kt} (\underbrace{G(t_0)}_{C} e^{-k t_0}) = C e^{k(t-t_0)}$$

Another application of Gronwall.

weak version of continuous dependence on (initial data)

Prop V.I Continuous dependence on initial data

Assume H_1 H_2

Let $x_1, x_2 \in C^1(\underbrace{[\tilde{t}, t_1]}_{I}, U)$ solve $\dot{x}_i = f(x_i)$ (not solve IVP)

$i = 1, 2.$

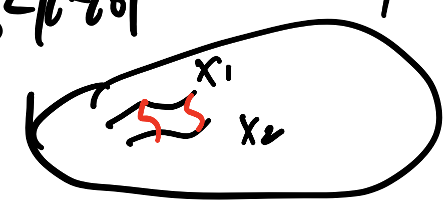
L: Lipschitz on $K := x_1(I) \cup x_2(I)$

of f compact $\rightarrow$ (continuous image of compact sets are compact)

For any $t_0 \in I$

$$|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)| e^{L|t-t_0|}$$

U

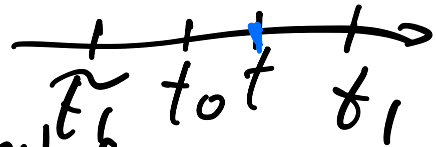


Proof: Define $h(t) := |x_1(t) - x_2(t)|$

Since $x_1(t) = x_1(t_0) + \int_{t_0}^{t} f(x_1(s)) ds$

$x_2(t) = x_2(t_0) + \int_{t_0}^{t} f(x_2(s)) ds$

For $t \in [t_0, t_1]$



$$h(t) = \left| x_1(t_0) - x_2(t_0) + \int_{t_0}^{t} f(x_1(s)) - f(x_2(s)) ds \right|$$

$$\leq |x_1(t_0) + x_2(t_0)| + \int_{t_0}^{t} |f(x_1(s)) - f(x_2(s))| ds$$

$$\le \underbrace{|x_1(t_0) + x_2(t_0)|}_{C} + \int_{t_0}^{t} \underset{K}{\underset{\downarrow}{L}}\, \underbrace{|x_1(s) - x_2(s)|}_{h(s)}\, ds$$

$$h(t) \le C + K \int_{t_0}^{t} h(s)\, ds$$

$\Rightarrow$ By applying Gronwall

$$h(t) = |x_1(t) - x_2(t)| \le Ce^{(t-t_0)}$$

for $t \in [\tilde{t}_1, t_0]$, repeat same proof using

$$x_1(t) = x_1(t_0) + \int_{t}^{t_0} [\, f(x_1(s))\, ds\,],$$

$$x_2(t) = x_2(t_0) + \int_{t}^{t_0} [\, -f(x_2(s))\, ds\,]$$

Necessity of hypotheses on f.

H1: f: continuous?

if discontinuous. Perfect thermostat example

$x(t)$: temperature in house

$x = 0$, outside.

$x = 1$ ideal temp.

$$\dot{x} = f(x) = \begin{cases} -x & \text{if } x > 1, \\ -x + 10 & \text{if } x \le 1 \end{cases}$$

heater is off

Newton's law

$\begin{cases} \text{no } \underline{\text{continuous}} \ \underline{\text{thermostat}}. \\ \underline{\text{piecewise}} \\ C^1 \text{ sol'n } \underline{\text{exist}} \end{cases}$ of cooling

initial, $x(0)=1$

H2. Local Lipschitz. (unique)

condensation on a raindrop:

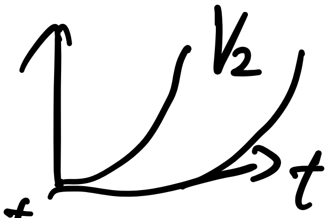
$V(t)$ volumn. grows proportionally to surface area.

$V:[0,\infty)\to\mathbb{R}$, $\dot V = |V|^{2/3}$ $\underline{V(0)=0}$

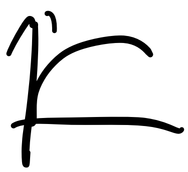
Sol'ns
$$\begin{cases} v_1(t)\equiv 0 \\ 1 = \dot v \cdot v^{-2/3} = \frac{1}{3}\left(v^{\frac{1}{3}}\right)' \end{cases}$$

$$t = \frac{1}{3} v_2(t)^{\frac{1}{3}} \to v_2(t) = \left(\frac{t}{3}\right)^3$$

$$V(t) = \begin{cases} 0 & t \le t_0 \\ \left(\frac{(t-t_0)}{3}\right)^3 & t > t_0 \end{cases}$$



No uniqueness. $|V|^{\frac{2}{3}}$ not Lipschitz



Rmk: Peano's existence theorem.

Finite existence time.

$$\dot{x} = x^2, \quad x(0) = x_0 > 0$$

$$f(x) = x^2$$

$$\dot{x} = x^2$$

$$1 = \dot{x} \cdot x^{-2} = -(x^{-1})'$$

$$\int \quad t = -\frac{1}{x(t)} + \frac{1}{x_0}$$

$$\Rightarrow \quad x(t) = \frac{1}{\frac{1}{x_0} + t} = \frac{x_0}{1 - x_0 t}$$

Sol'n exist on $\left(-\infty, \frac{1}{x_0}\right)$

explode.. $t \geq \frac{1}{x_0}$

Lec 4

H1: $f: U \to \mathbb{R}^n$ continuous, $U \subset \mathbb{R}^n$ open

H2. f is locally Lipschitz. $\forall K \subset U$ compact

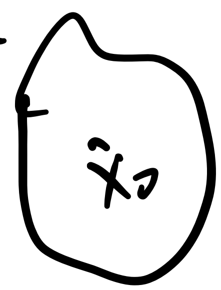
$\exists L > 0$, $|f(p) - f(q)| \le L|p - q|$. $\forall p, q \in K$

Thm. Assume

f satisfies H1, H2.

Fix $x_0 \in U$. $\exists a > 0$ s.t. the IVP

U



$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$$

$\dot{x}(t) = f(x(t))$ $\forall t \in I$.

$\dot{x} = \dfrac{dx(t)}{dt}$.

has a unique sol'n on $[t_0 - a, t_0 + a]$

x solves IVP iff ($\Leftrightarrow$) x continuous and sat

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))\,ds$$

$\Leftrightarrow$ x is cts fixed point of $F(y)(t)$

$$= x_0 + \int_{t_0}^{t} f(y(s))ds$$

Thm. Banach Fixed Point Thm.

(X,d) (complete). non-empty metric space and

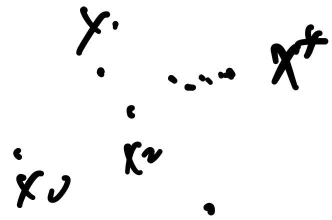
$F: X \to X$ contraction mapping

$\exists K \in [0,1)$, s.t. $d(F(x), F(y)) \leq K d(x,y), \forall x,y \in X$

Then $\exists!\ x^* \in X$ s.t. $F(x^*) = x^*$

Pf: Fixed x_0, let $x_1 = F(x_0)$

$x_k = F(x_{k-1})$



$d_j = d(x_j, x_{j+1}) \leq K\, d(x_{j-1}, x_j) \leq \cdots \leq K^j d_0$

Claim. $\{x_j\}$ Cauchy

Fix $i \leq j$, $d(x_i, x_j) \leq d(x_i, x_{i+1}) + \cdots + d(x_{j-1}, x_j)$

$d_{i+1} + \cdots + d_j$

$\leq d_0 \sum_{k=i}^{j-1} K^k$

$\leq d_0 K^i \frac{1}{1-K}$

$i \to \infty$

$d_{ij} \to 0$.

So. $\exists x^*$ s.t. $d(x_j, x^*) \to 0$. (By complete. space)

$x_j \to x^*$.

On one hand

OTOH.

$$d(F(x_j), F(x^*)) \le K\, d(x_j, x^*) \xrightarrow{j\to\infty} 0$$

$|F(x_j) \xrightarrow{j\to\infty} F(x^*)|$

$\Rightarrow F(x^*) = x^* \Leftrightarrow x^*$ fixed point

limit is unique.

Uniqueness. Let x_1 x_2 be fixed pt.

$$\underline{d(x_1, x_2)} = d(F(x_1), F(x_2)) \le K\, \underline{d(x_1, x_2)}$$

$= 0$ //

§. Observation. Assume H_1, H_2 ✓

If x_1, x_2 solve.

$$\begin{cases} \dot{x}_i = f(x_i) \\ x_i(t_0) = x_0 \end{cases}$$

on compact intervals I_1, I_2

Then $\underline{x_1 = x_2 \text{ on } I_1 \cap I_2}$

Indeed

$$|x_1(t) - x_2(t)| \le \underbrace{|x_1(t_0) - x_2(t_0)|}_{=0} e^{L|t-t_0|}$$

By weak initial dependance.

Lipschitz.

$x_1(I_1) \cup x_2(I_2)$

Given f, x_0, x sol'n to $\begin{cases}\dot{x}=f(x)\\ x(t_0)=x_0\end{cases}$ (may be bad)

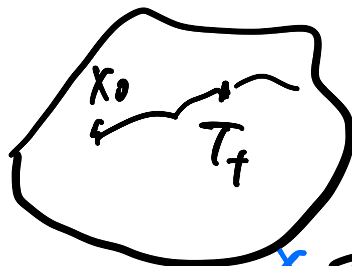
e.g. $T_+<\infty$
$\dot{x}=x^2$, $x(0)=x_0>0$
$T_+=\frac{1}{x_0}$. $\Rightarrow x=\frac{x_0}{1-x_0 t}$
$x(T_+)=\infty$ non-exist

$T_+ := \sup\{t_1>t_0 :$ the sol'n $x\in C^1([t_0,t_1];U)$ to IVP exists$\}$

Maximal forward time of existence

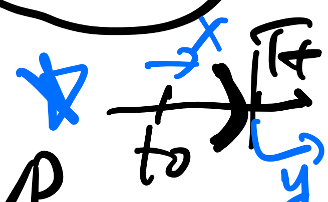
If $T_+<\infty$ (finite)

then there is no extension.



$\tilde{x}\in C^1([t_0,T_+];U)$

Otherwise, we could extend x even farther by solving IVP



(new IVP)

$$\begin{cases}\dot{y}=f(y)\\ y(T_+)=x(T_+)\end{cases}$$

$T_- := \{t_1<t_0 :$ the sol'n $x\in C^1([t_1,t_0],U)$ to IVP exist$\}$

maximal backward time of existence.

$\boxed{(T_-,T_+)}$ maximal interval of existence.

Q: If $T_+<\infty$, what happens to x as $t\to T_+$?

Thm. Let f sat H_1 H_2.

x be sol'n $\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$ on $[t_0, T_+)$

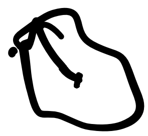
If $T_+ < \infty$, then $\forall K \subset U$ compact.

$\exists \tau > 0$, s.t. $x(t) \notin K$ $\forall t \in [T_+ - \tau, T_+)$

(other than) .

if not → toward boundary ✓.

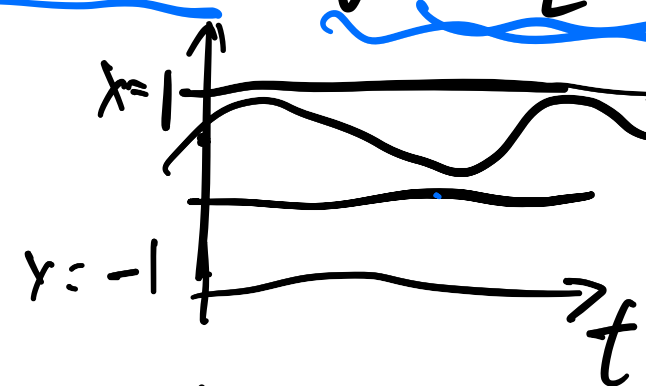
Contrapositive.



If $\exists K \subset U$ compact such that

$x(t) \in K$ $\forall t \in [t_0, T_+)$, then

$T_+ = +\infty$



barrier?

M1. 9/30 or 10/7

M2. 11/11 or 11/18

Final: Last week

Forward invariance?

existence of CBF?

if T_+ (in U)

$x(t) \in K \subset U$

$\Rightarrow$ K is FI. ($t \to \infty$)

Lec 5

Recall $T_+ := \{\sup t_1 > t_0 :$ the sol'n $x \in C^1([t_0, t_1], U)$ to IVP exists.$\}$

$f: U \to \mathbb{R}^n$ $\qquad H_1: f$: cont.

$U \subset \mathbb{R}^n$ open. $\qquad H_2: f$. locally Lipschitz.

IVP: $\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$

Thm Let f sat. H_1, H_2. Let x solve (IVP) on $[t_0, T_+)$

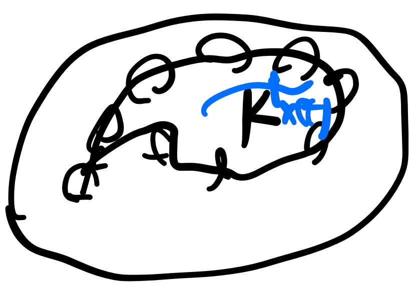
If $T_+ < +\infty$, then $\forall K \subset U$ compact

$\exists \tau > 0$, s.t. $x(t) \notin K$. $\forall t \in [T_+ - \tau, T_+)$

($\Leftrightarrow$ $\forall t \geq T_+ - \tau \Rightarrow x(t) \notin K$
$x \in K \Rightarrow t < T_+ - \tau$)

Pf: Fix $K \subset U$ compact

U



$K_r = \{x \in \mathbb{R}^n : dist(x, K) \leq r\}$ y_0

$\subset U$. K_r compact.

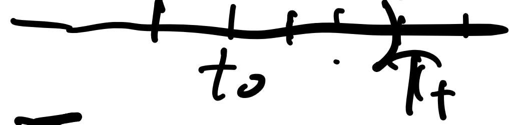
Let $M = \sup_{q \in K_r} |f(q)|$, L. Lip constant for f valid on K_r

Let $\tau = \frac{1}{2} \min\left\{\frac{1}{L}, \frac{r}{M}\right\}$

why?

$t - \tau$ OK for y if not exist

t $\boxed{t+\tau}$ $t+\tau$



Take t s.t. $x(t) \in K$

Want to show $t < T_+ - \tau$

By $\exists$ Thm, $\exists\, y \in C^1([t-\tau, t+\tau], U)$ s.t. $\begin{cases} \dot{y} = f(y) \\ y(t) = x(t) \end{cases}$ y_0

By ! part, $y = x$ on interval where both exist

If $t + \tau \geq T_+$, recall we cannot extend sol'n x to C^1 fn on $[t_0, T_+]$ (x not exist)

So we must have $t + \tau < T_+$ //

Ex.1. Sps $f: \mathbb{R}^n \to \mathbb{R}^n$ loc. Lip and bounded (by $M > 0$). Then for any $x_0 \in \mathbb{R}^n$, the sol'n to $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ exists for all $t \in \mathbb{R}$

Pf. Let's show $T_+ = +\infty$

Suppose $T_+ < +\infty$

Then $\forall\, t \in [0, T_+)$

$$x(t) = x_0 + \int_0^t f(x(s))\,ds$$

$$\Rightarrow |x(t) - x_0| \leq \int^t |f(x(s))|\,ds$$

$\le Mt \le MT_+$

$\Rightarrow x(t) \in \overline{B(x_0, MT_+)}$ closed ball compact.

But by previous thm,
$x(t)$ must leave every compact set as $t \to T_+$. $\to\Leftarrow$ //.

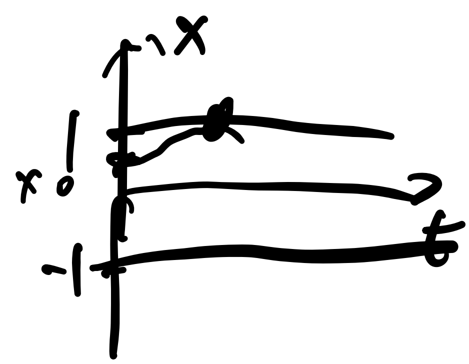
Ex. 2. $\begin{cases} \dot{x} = 1 - x^2 \\ x(0) = x_0 \in (-1, 1) \end{cases}$

$f: \mathbb{R} \to \mathbb{R}$ cont.

$f(r) = 1 - r^2$ ✓

locally Lip ✓.

Claim.
Sol'n x exists for all $t \in \mathbb{R}$



Pf: Obs. $y_+(t) \equiv 1$, $y_-(t) \equiv -1$ solve $\dot{x} = 1 - x^2$

$x_0 = +1$ $\quad$ $x_0 = -1$

So for $t \in (T_-, T_+)$, $x(t) \ne \pm 1$ by uniqueness.

if $x(t) = 1$, $x = y$ $\Rightarrow 1 = x_0$ $\quad$ $x_0 \ne 1$

$y_+ = y_- = x$ over $(0,t)$

if $\exists t_1, x(t_1) > 1$
$x_0 = x(t_0) < 1 < x(t_1)$
$\Rightarrow \exists t', x(t') = 1$

So by intermediate value thm.

$$x(t) \in (-1,1) \subset [-1,1]$$

Since $x(t)$ trapped in compact set.
by previous thm, $(T_-, T_+) = (-\infty, +\infty)$

Thm. Continuous dependence on initial data
pt 2).

assume (H1). (H2).

and let $x, \hat{x}$ be solutions to

$$\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}, \quad \begin{cases} \dot{\hat{x}} = f(\hat{x}) \\ \hat{x}(t_0) = \hat{x}_0 \end{cases}.$$

on maximal forward interval of existences.

$[t_0, T_+(x_0))$

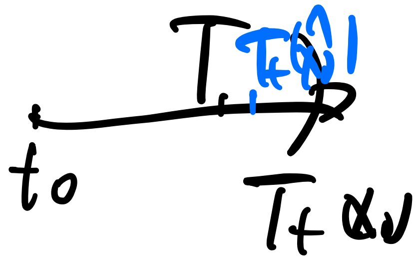
$[t_0, T_+(\hat{x}_0))$

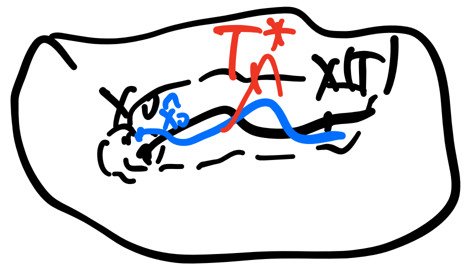
Then. $\forall T \in [t_0, T_+(x_0))$ and $\forall \varepsilon > 0$,
$\exists \delta > 0$, s.t. if $|x_0 - \hat{x}_0| < \delta$ then

. $T_f(\hat{x}_0) > T$

. $\forall t\in[t_0,T], \ |x(t)-\hat{x}(t)|<\varepsilon$ bandwidth is ε

2ε





Proof Fix $T\in[t_0,T_f(x_0))$, fix $\varepsilon>0$

Let $K=K_{T,\varepsilon}=\{q\in\mathbb{R}^n : \mathrm{dist}(q, x([t_0,T]))\le\varepsilon\}$

wlog $\varepsilon\ll1$ s.t. $K\subset U$

Let $L=L_{T,\varepsilon}$ be Lip constant for f valid on K

Let $\delta=\frac{\varepsilon}{2}e^{-L(T-t_0)}$

Let $|x_0-\hat{x}_0|<\delta$

If $t_*\in[t_0,T]$ is s.t. $\hat{x}([t_0,t_*])\subset K$

then from the weak version on Lec 3.

$$|x(t)-\hat{x}(t)|\le|x_0-\hat{x}_0|e^{L(t-t_0)}$$

$L: x([t_0,T])\cup\hat{x}([t_0,t_*])$

$$<\delta \quad \forall t\in[t_0,t_*]$$

$$<\frac{\varepsilon}{2}$$

if $t_* = t_0$, this holds

Let $T_* = \sup\{t_* \geq t_0 : \underbrace{\hat{x}([t_0, t_*])}_{\text{compact}} \subset K\}$

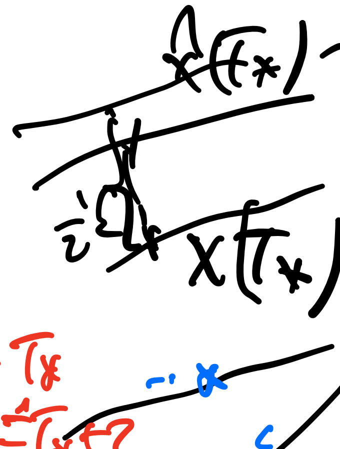
Note $\hat{x}([t_0, T_*]) \subset K$. So

$$|x(t) - \hat{x}(t)| < \frac{\varepsilon}{2} \quad \forall t \in [t_0, T_*]$$

similar

Claim. $T_* \geq T$

Proof: Suppose $T_* < T$



why? By continuity

$T_* < T$. Put T_* as t_* in $\triangle$ $\begin{cases} t_0 = T_* \\ t_* = T_* + \tau \end{cases}$

$\exists\, \tau > 0$ s.t.

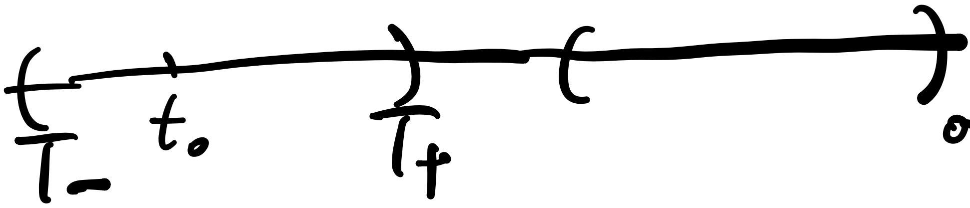
$$|x(t) - \hat{x}(t)| < \varepsilon \text{ on } [t_0, T_* + \tau]$$

(T_*)

$\hat{x}(t) \in K$

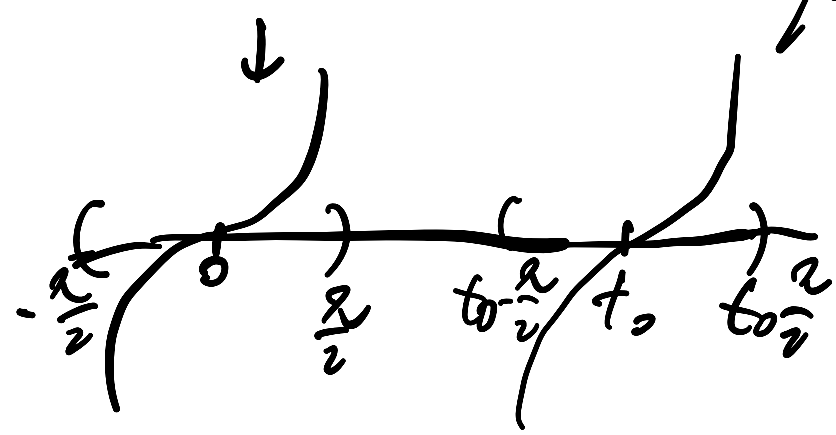
contradict to T_* as a sup //.

Lec 6.



other $((\cdot\cdot T_+))$ ↓ exist but not unique

Ex. $\begin{cases} \dot{x} = 1 + x^2 \\ x(0) = 0 \end{cases}$ $\quad x(t) = \tan(t)$



Recall. Thm. (Continuous dependence on initial data)

~~Assume~~ $H_1 . H_2$

Fix $x_0 \in U$ and let $T_+(x_0)$ be max forward time of existence for sol'n to $\begin{cases} \dot{x} = f(x) \\ x(t_0) = x_0 \end{cases}$

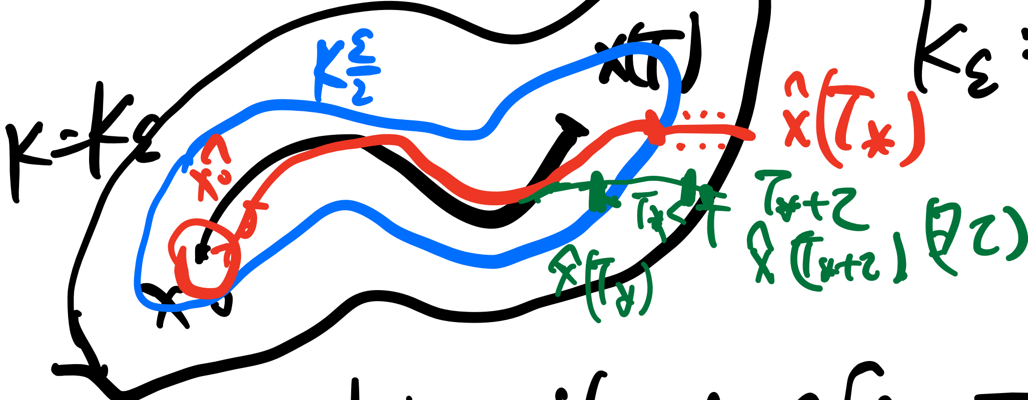
Then $\forall \varepsilon > 0$, $\forall T \in [0, T(x_0))$,

$\exists \delta > 0$, s.t. if $|\underset{\in U}{\hat{x}_0} - x_0| < \delta$ $(\forall \hat{x}_0)$ $\quad \delta \sim \varepsilon, T, x_0.$

Then $\cdot\ T_+(\hat{x}_0) \geq T$

$\cdot\ |x(t) - \hat{x}(t)| < \varepsilon, \forall t \in [0, T]$

Proof cont'd



$K_\varepsilon := \{ q : \mathrm{dist}(q, x([t_0,T])) \le \varepsilon \}$

Aux claim (By last lec): if $t_* \in [t_0, T]$ s.t. $\hat{x}$ exists on $[t_0, t_*]$

$\hat{x}([t_0, t_*]) \subset K_\varepsilon$ then $\exists\, \delta = \frac{\varepsilon}{2} e^{-L(T-t_0)}$ (L: Lip. over K_ε.)

$\hat{x}([t_0, t_*]) \subset K^{\frac{\varepsilon}{2}}$

Let $T_* = \sup\{ t_* \ge t_0,\ \hat{x}$ exists on $[t_0, t_*]$ and $\hat{x}([t_0, t_*]) \subset K_\varepsilon \}$

- t_0 is in T_*
- $\forall \tau > 0$, $\hat{x}([0, T_* + \tau]) \not\subset K_\varepsilon$ by definition

By Aux claim. $\hat{x}([t_0, T_*]) \subset K_{\varepsilon/2}$

$\hat{x}$ is cont.: $\forall \varepsilon^*\ \exists \tau$ s.t. $\forall t \in [T_*, T_*+\tau]$ $|\hat{x}(t) - \hat{x}(T_*)| < \varepsilon^* < \frac{\varepsilon}{2}$.

By continuity. $\exists\, \tau > 0$, s.t. $\hat{x}(t) \in K_\varepsilon$

$\forall t \in [T_*, T_* + \tau]$

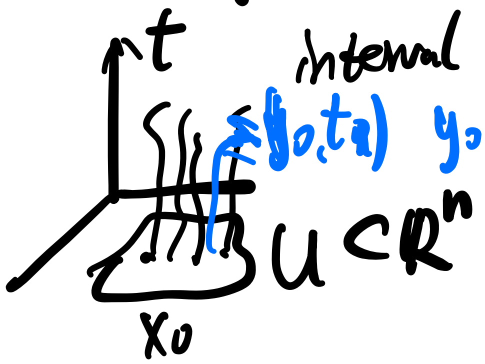
thus $\forall t \in [t_0, T_* + \tau]$

If $T_* < T$ (T_* can be extended.) $\to$ Contradict with def. of T_*.

So $T_* \ge T$. //

(sup).

Flow of an ODE system.



$y(0,t_1)$ $y_0 = x_0 \to \in U$.

By existences + uniqueness, these integral curves don't intersect

Notation for $x_0 \in U$.

let x solve IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$ and

let $(T_-(x_0), T_+(x_0))$ be max interval of existence. For t, let

$$\phi(t, x_0) = x(t)$$

Set $U_T = \begin{cases} \{x_0 \in U : T_+(x_0) > T\} & \text{if } T \geq 0 \\ \{x_0 \in U, T_-(x_0) < T\} & \text{if } T \leq 0 \end{cases}$

max interval of existence $> T$

So $\phi : [0, T] \times U_T \to U$ if $T > 0$

$[T, 0] \times U_T$ if $T < 0$

$\phi(T, x_0) = x(T)$
$x(0) = x_0$

Rmk. for each $T\in\mathbb{R}$. U_T is open.

by continuous dep on initial data

small enough.

$\exists\ |x(t)-x|<\varepsilon$. $\exists\ \delta\ |y-x_0|<\delta$

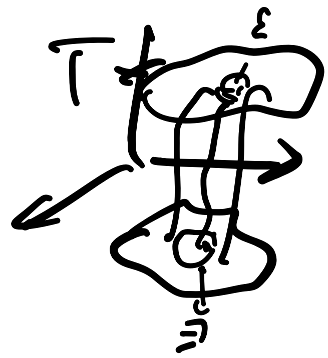
$T_f \in U_T$. st $T_f(x_0) > T$.

U.

Def. The time T flow map of the ODE system (or of the vector field f)

is $\phi_T : U_T \to U$

$\phi_T(x_0) = \phi(T, x_0) = x(T)$ (with $x(0)=x_0$) ?



Properties.

$\phi_0 : U \to U$ is identity map

Suppose U_T non-empty. $x_0 \to x(T)$

· ϕ_T is continuous. by continuous dependence on init data

· ϕ_T is injective. uniqueness of ϕ_T.

So $\phi_T : U_T \to \phi_T(U_T)$ is bijective.

Claim. $\phi_T^{-1} = \phi_{-T}$ on $\phi_T(U_T) = U_{-T}$

Fix $y_0 \in \phi_T(U_T)$, let $x_0 = \phi_T^{-1}(y_0)$

I.e. for sol'n x to IVP $\begin{cases}\dot{x} = f(x)\\ x(0) = x_0\end{cases}$

$x(T) = y_0$

$y_0 = x(T)$

$x_0 = x(0)$

x also solves IVP $\begin{cases}\dot{x} = f(x)\\ x(T) = y_0\end{cases}$

x exists on $[0, T]$

Let $\tilde{x}(t) = x(t+T)$ solves $\begin{cases}\dot{\tilde{x}} = f(\tilde{x})\\ \tilde{x}(0) = y_0\end{cases}$

and exists on $[-T, 0]$

$\tilde{x}(-T) = x_0$

i.e. $x_0 = \phi_{-T}(y_0)$

$x_0 = \phi_T^{-1}(y_0)$

$\Rightarrow \phi_T(U_T) \subset U_{-T}$

$y_0 \in$

$\Rightarrow y_0 \in U_{-T}$

Lec 7

Recall. Assuming H_1, H_2.

For $x_0 \in U$, let $x(t)$ be maximal solution to $\begin{cases}\dot{x} = f(x)\\ x(0)=x_0\end{cases}$

on $(T_-(x_0), T_+(x_0))$

on $U_T = \begin{cases} \{x_0\in U: T_+(x_0)>T\} & \text{if } T\geq 0\\ \{x_0 \in U, T_-(x_0)<T\} & \text{if } T\leq 0\end{cases}$

Let $\Phi(T,x_0) = \phi_T(x_0) = x(t)$

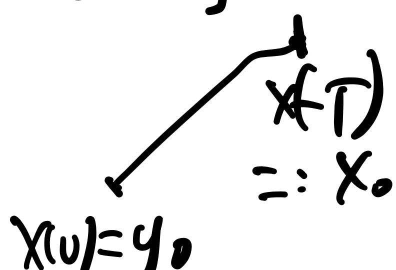
If $U_T\neq\phi$, Φ_T continuous, $U_T \to \Phi_T(U_T)$ bijective

and on $\Phi_T(U_T)$, $\Phi_T^{-1} = \Phi_{-T}$ $\quad \forall y_0\in\Phi_T(U_T) \Rightarrow y_0 \in U_{-T}$

Claim $\Phi_T(U_T) = U_{-T}$ $\quad$ have shown. $\Phi_T(U_T)\subset U_{-T}$

Now let $y_0\in U_{-T}$ x exists on $[-T, 0]$

Let x solve $\begin{cases}\dot{x} = f(x)\\ x(0)=y_0\end{cases}$



so x solves $\begin{cases}\dot{x} = f(x)\\ x(-T)=x_0\end{cases}$

Let $\tilde{x}(t) := x(t-T)$

solves $\begin{cases} \dot{\tilde{x}} = f(\tilde{x}) \\ \tilde{x}(0) = x_0 \end{cases}$

. $\tilde{x}$ exists on $[0,T]$ and $\tilde{x}(T) = y_0$

i.e. $x_0 \in U_T$ and $\Phi_T(x_0) = y_0$

So $y_0 \in \Phi_T(U_T)$

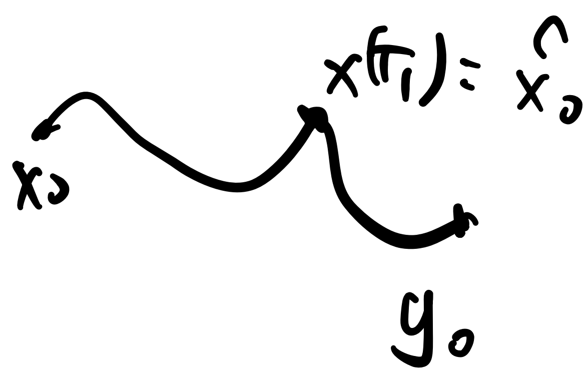
Thm. If $U_T \neq \phi$, then $\Phi_T : U_T \to U_{-T}$ is a homeomorphism (bijection. continuous inverse itself.)

Thm. (Semigroup property)

Fix $T_1, T_2 \geq 0$, then $\Phi_{T_1+T_2} = \Phi_{T_2} \circ \Phi_{T_1}$ on

$\tilde{U} = \{ x_0 \in U_{T_1} : \underline{\Phi_{T_1}(x_0) \in U_{T_2}} \} \supseteq U_{T_1+T_2}$

Proof. Fix $x_0 \in \tilde{U}$,



$\hat{x}_0 = \Phi_{T_1}(x_0)$

$y_0 := \Phi_{T_2}(\hat{x}_0)$

we want to show

$$T_+(x_0) > T_1 + T_2 \text{ , and } \Phi_{T_1+T_2}(x_0) = y_0$$

Let x solve $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$, exists on $[0, T_1]$

Set $\hat{x}(t) = x(t+T_1)$ solves $\begin{cases} \dot{\hat{x}} = f(\hat{x}) \\ \hat{x}(0) = \hat{x}_0 \end{cases}$

exists on $[-T_1, 0]$

By $\hat{x}_0 \in U_{T_2}$, $\hat{x}_0$ exists on $[0, T_2]$

$\Rightarrow \hat{x}_0$ exists on $[-T_1, T_2]$

So x exists on $[0, T_1+T_2)$.

$(x(t) = \hat{x}(t-T_1))$ and $\Phi_{T_1+T_2}(x_0) = x(T_1+T_2)$

$= \hat{x}(T_2) = \Phi_{T_2}(\hat{x}_0) = y_0$

$= \Phi_{T_2} \circ \Phi_{T_1}(x_0)$

So $x \in \tilde{U} \Rightarrow x \in U_{T_1+T_2}$.

($\Leftarrow$ similarly.)

Thm. Assume $f: U \to \mathbb{R}^n$ is C^1

$\subset \mathbb{R}^n$ open.

(C_1 includes $\begin{cases} H_1 \\ H_2 \end{cases}$)

Then $\forall T \in \mathbb{R}$ s.t. $U_T \neq \phi$

$x_0 \mapsto \Phi_T(x_0)$ is $C^1(U)$

Aside, Thm contin dep on initial data.

Let f satisfy H_1, H_2. Fix $\varepsilon > 0$, $T \geq 0$, $x_0 \in U_T$. Then $\exists \delta = \delta(\varepsilon, T, x_0)$ s.t. if $|\hat{x}_0 - x_0| < \delta$, then

① $\hat{x}_0 \in U_T$

② $|x(t) - \hat{x}(t)| < \varepsilon$, $\forall t \in [0, T]$

$\updownarrow$ can be worse at the boundary.. depends on x_0

Proof. Formal Comp.

Fix $e \in S^{n-1} \subset \mathbb{R}^n$, What is $\frac{\partial \Phi}{\partial e}(T, x_0)$?

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial e}(t, x_0)\right) = \frac{\partial}{\partial e}\left(\frac{\partial \Phi}{\partial t}(t, x_0)\right)$$

$$= \frac{\partial}{\partial e} f(\Phi(t, x_0)) \qquad \frac{dx(t)}{dt} = f(x)$$

$$= \underbrace{Df(\Phi(t, x_0))}_{\text{matrix } A} \cdot \underbrace{\frac{\partial \Phi}{\partial e}(t, x_0)}_{u}$$

$f = (f_1 \cdots f_n)$ n.

$$\underset{n \times n}{Df} = \begin{pmatrix} \nabla f^1 \\ \vdots \\ \nabla f^n \end{pmatrix} = (\partial_1 f \cdots \partial_n f)$$

Set $A(t) = Df(\Phi(t, x_0))$, $f \in C^1$.

$u(t) = \frac{\partial \Phi}{\partial e}(t, x_0)$

continuous

u solves $\begin{cases} \dot{u}(t) = A(t)\, u(t) \\ u(0) = e \end{cases}$? $u(0) = I$ $\quad$ $u = e^{t}$ $\quad$ $u(0) = I$ $\quad$ $\frac{\partial u}{\partial t} = I \cdot u$

$\Phi(0, x) = x$ $\quad \therefore = e^{0}$

$D\Phi(0, x) = I$

$D\Phi(0, x)\, e = e$

By HW, let $A : [0, T_{+}(x_0)) \to \mathbb{R}^{n \times n}$

be the continuous fn def by

$$A(t) = Df(\phi(t, x_0))$$

For each $k = 1, \dots n$. $\exists!$ sol'n to

$$\begin{cases} u_k'(t) = A(t)\, u_k(t) \\ u_k(0) = e_k \end{cases}$$

Lec 8.

Thm. Assume $f: U \to \mathbb{R}^n$. C^1 (open $\subset \mathbb{R}^n$)

Then. $\forall T \in \mathbb{R}$. s.t. $U_T \neq \emptyset$

$\underbrace{\text{map}: x_0 \mapsto \phi_T(x_0)}$ is C^1

Last time. Formal comp

We expect $\partial_{e_k} \Phi_T(x_0) = u_k(T)$ e_k: base

where $\begin{cases} \dot{u}_k(t) = A(t)\, u_k(t),\ A(t) = Df(\phi_t(x_0)) \\ u_k(0) = e_k \end{cases}$ (fixed other $1 \cdots n$, $i \neq k$)

existence & uniqueness by HW3.

Let $L_T: \mathbb{R}^n \to \mathbb{R}^n$ be def by

$\dot{L}_T(y)$

x_T 对 e_k 的 grad. $L_T(y) = (u_1 | \cdots | u_n) \cdot y$ each col is $u_k(T)$

$D(f_1 \cdots f_n) = (\partial f_1, \partial f_1 \cdots \partial f_n)$

x_0 对 e_k 的 grad 为 y

Expect. $D\Phi_T(x_0) = L_T$

obs $v(t) = L_t(y)$ solves $\begin{cases} \dot{v}(t) = A(t)\, v(t) \\ v(0) = y \end{cases}$

$L_0(y) = y$

y is the deviation from x_0

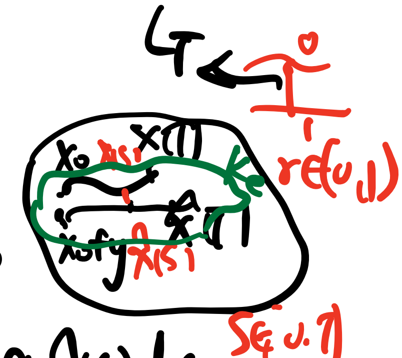
Prop. A Fix $T \in (0, T_+(x))$ Then

$$\frac{|\phi_T(x_0+y) - \phi_T(x_0) - L_T(y)|}{|y|} \to 0 \quad \text{as } |y| \to 0$$

$\to R_T(y)$

Pf: Step 0. For $t \in [0,T]$, let

$$x(t) = \Phi_t(x_0) = x_0 + \int_0^t f(x(s))\,ds$$



$\hat{x}(t) = \Phi_t(x_0+y) = x_0 + y + \int_0^t f(\hat{x}(s))ds$ $\hat{x}(s) = \{v(s)+R_s(y)\}r$

$v(t) = y + \int_0^t A(s)v(s)ds$ ★

$v(t) = L_T(y)$

$R_t(y) = \int_0^t f(\hat{x}(s)) - f(x(s)) - A(s)v(s)\, ds$

we want to show $\frac{|R_t(y)|}{|y|} \to 0$ as $|y| \to 0$

$\left(\frac{\partial f}{\partial r}\right) = Df\frac{\partial x}{\partial r}$

Step 1. Est $R_t(y)$ + use Gronwall

F.Th. Calculus: $f(\hat{x}(s)) - f(x(s)) = \underbrace{\int_0^1 Df(\gamma^s(r))dr}_{B(s)}$ $\times\underbrace{\{v(s)+R_s(y)\}}_{\dot{\gamma}^s(r)}$

$\Rightarrow R_t(y) = \int_0^t B(s)R_s(y) + \{B(s)-A(s)\}v(s)ds$

$Df(x(s)) = Df(\gamma^s(0))$

$\Rightarrow |R_t(y)| \leq \underline{\lambda(y)\int_0^t R_s(y)ds}$ $\underline{+ M(y)t}$

$\lambda(y) = \sup_{s\in[0,T]} \|B(s)\|$

$M(y) = \sup_{s\in[0,T]} |\{B(s)-A(s)\}v(s)|$

not related to t

Gronwall. $|R_T(y)| \leq M(y)T e^{\lambda(y)T}$

want to show. $\frac{M(y)}{|y|} T e^{\lambda(y)T} \to 0$ as $|y| \to 0$

must decay

$\lambda(y) \geq 0$.

can not decay too much

Fix $\varepsilon > 0$, want $\delta > 0$ s.t. if $|y| < \delta$, $\frac{M(y)}{|y|} T e^{\alpha(y)T} < \varepsilon$

Step 2. Estimate $\alpha(y)$

Let ρ_1 be small enough. s.t.

$B(x_0, \rho_1) \subset U_T$ (Cont. dep on initial data)

$$K_\rho = \{ q \in \mathbb{R}^n, \operatorname{dist}(q, x[0,T]) \le \rho \} \subset U$$

$\forall \rho \le \rho_1$

Fix $\rho \le \rho_1$, Contin dep. on initial data.

$\Rightarrow \exists \delta_\rho > 0$, s.t. if $|y| < \delta_\rho$, $\hat{x}(t) \in K_\rho$ $\forall t \in [0,T]$

$$\alpha(y) \le \sup \{ \|\underbrace{Df(q)}_{\text{continuous}}\| : q \in \underbrace{K_\rho \}}_{\text{compact}} =: C_1$$

$\alpha(y) = \sup_{s \in [0,T]} \int_0^1 Df(\gamma^s(r)) dr$

$\le \sup_{s \in [0,T]} \sup_{r \in [0,1]} |Df(\gamma^s(r))|$

$= \sup \{ Df(q) : q = \gamma^s(r), r \in [0,1], s \in [0,T] \}$

$\le \sup \{ Df(q) : q \in K_\rho \}$

Upshot if $\rho \le \rho_1$, $|y| < \delta_\rho$ then.

$$\frac{R_T(y)}{|y|} \overset{\text{Gronwall}}{\le} \frac{M(y)}{|y|} T e^{\alpha y T}$$

$$\le \frac{M(y)}{|y|} \bar{C}_1$$

$\bar{C}_1 = T e^{C_1 T}$

Lec. 9.

Thm. Assume $f: U \to \mathbb{R}^n$ C^1. Then $\forall T \in \mathbb{R}$
$U \subset \mathbb{R}^n$ op

s.t. $U_T \neq \phi$, $x_0 \mapsto \phi_T(x_0)$ is C^1

Formal computation → expect $D\phi(T, x_0)$

$$= L_T \to \text{linear function}$$

$$L_T: \mathbb{R}^n \to \mathbb{R}^n,$$

$$L_T(y) = \big(u_1(T) \,|\, \cdots \,|\, u_n(T)\big)\, y$$

where $\begin{cases} \dot u_k(t) = Df(\phi_t(x_0))\, u_k(t) \\ u_k(0) = e_k \end{cases}$

Prop. A. Fix $x_0 \in U$, $T \in (0, T_+(x_0))$. Then

$$\frac{|R_T(y)|}{|y|} = \frac{|\phi_T(x_0+y) - \phi_T(x_0) - L_T(y)|}{|y|} \to 0$$

direction derivative of $\phi_T(\cdot)|_{x_0}$ is L_T

linear interpolate $\gamma^s(r)$

$K_\epsilon = \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, x(t)) \le \epsilon\}$

We showed if $|y| < \delta_\epsilon$

$$\frac{|R_T(y)|}{|y|} \le \frac{c_1}{|y|} M(y)$$

$$M(y) = \sup_{s \in [0,T]} \left| \int_0^1 \{ Df(\gamma^s(r)) - Df(x(s)) \}\, dr\, L_s(y) \right|$$

$\gamma^s(0)$

$$\le \underbrace{\sup_{s \in [0,T]} \int_0^1 |Df(\gamma^s(r)) - Df(x(s))|\, dr}_{*} \times \sup_{s \in [0,T]} |L_s(y)|$$

$$|L_s(y)| = |(u_1(s) | \cdots | u_n(s)) \cdot y|$$

$$\le n^2 \cdot \sup_{i,j=1\cdots n} |u_i(s)e_j| \cdot |y| \le \bar{C}_2 |y|$$

subset of K_e. upper bound of $\|\ \|$ operator norm

only dep on f, x_0, T. (not y)

$$* \le \sup_{s\in[0,T]} \sup_{q\in \bar{B}(x(s),\rho)} \|Df(q) - Df(x(s))\| \le \frac{\varepsilon}{\bar{C}_1 \bar{C}_2}$$

choose $\rho \le \rho_1$ small enough. so

(and accordingly take $|y| < \delta_\rho$

Summary $|y| < \delta_\rho \Rightarrow \hat{x}(t) \in K_\rho$ for $\hat{x}(t) = \phi_t(x_0+y)$

So $M|y| \le \bar{C}_2 |y| \cdot \frac{\varepsilon}{\bar{C}_1 \bar{C}_2}$ and $\frac{|R_T(y)|}{|y|} \le \bar{C}_1 \frac{M|y|}{|y|} \le \varepsilon$

Prop B. $x_0 \mapsto D\Phi(T, x_0)$ is continuous

Proof. $D\Phi(T, x_0) = \left(u_1^{x_0}(T) \mid \cdots \mid u_n^{x_0}(T)\right)$

$$\begin{cases} \dot{u}_k^{x_0}(t) = \underbrace{Df(\phi(t,x_0))}_{A^{x_0}(t)} u_k(t) \\ u_k(0) = e_k \end{cases}$$

(WLOG. $x_0 \to u_1^{x_0}(T)$ cts)

Fix $\varepsilon > 0$, Let $\delta =$ ______ For $|\hat{x}_0 - x_0| < \delta$,

Let $w(t) = u_1^{x_0}(t) - u_1^{\hat{x}_0}(t)$. $w(t)$ solves

$$(**) \begin{cases} \dot{w}(t) = A^{x_0}(t) u_1^{x_0}(t) - A^{\hat{x}_0}(t) u_1^{\hat{x}_0}(t) \\ \quad = A^{x_0}(t) w(t) + \underbrace{\left[A^{x_0}(t) - A^{\hat{x}_0}(t)\right]}_{*} u_1^{\hat{x}_0}(t) \\ \dot{w}(0) = 0 \end{cases}$$

we want to show $|w(T)| < \varepsilon$

$$* = Df(\hat{\Phi}(t, x_0)) - Df(\hat{\Phi}(t, \hat{x}_0))$$

Fix $\varepsilon' > 0$, choose $\rho > 0$, small enough

$\sup_{t \in [0,T]} \sup_{q \in B(x(s), \rho)} |Df(q) - Df(x(t))| < \varepsilon'$ (equ.1) $\to$ Df is contiuous ($f \in C^1$)

Choose $\delta > 0$, so that if $\delta \le \delta'$, $\Phi(t, \hat{x}_0) \in K_\rho$ (cnt. dep initial data) $\forall t \in [0,T]$

(depon ε' ρ)

$\forall \rho$, choose δ $|*| \le (\text{equ.1})$; $\forall \varepsilon'$, choose ρ, $(\text{equ.1}) < \varepsilon'$ $\Rightarrow$ $\forall \varepsilon'$, $|*| < \text{equ.1} < \varepsilon'$ choose δ

$\Rightarrow |*| < \varepsilon'$.

solve(**)

$$|w(t)| \le 0 + \int_0^t \underbrace{|A^{x_0}(s)|}_{\le C}\, |w(s)|\, ds + \int_0^t \underbrace{|*_s|}_{<\varepsilon'}\, \underbrace{|u_1^{\hat{x}_0}(s)|}_{\le C_2}\, ds$$

cnt over compact. cnt over compct.

$$\le C_1 \int_0^t |w(s)|\, ds + \varepsilon' C_2$$

$\Rightarrow$ By Gronwall. $\forall t \in [0,T]$

$$|w(T)| \le \varepsilon' C_2 e^{C_1 T} < \varepsilon$$

hold. choose ε' s.t..

choose ρ s.th $|*|$

PDE Laplace Equation.

$U \subset \mathbb{R}^n$ open $u : \bar{U} \to \mathbb{R}$

Laplacian $\Delta u(x) = \sum_{i=1}^n \partial_{x_i} \partial_{x_i} u(x)$