21-632 Class I: Intro and overview.

ODE: a few weeks.
PDE: main focus
8 hws.

Differential equation

$$F(D^k u(x), \dots, Du(x), u(x), x) = 0$$

F ↓ unknown (k-th order derivative) with initial/boundary condition

no general theory to solve all DEs

cannot solve explicitly, develop methods for analyzing them

PDE.

first-order

$$F(Du(x), u(x), x) = 0$$

transport equation.

$b \in \mathbb{R}^n$, $u: \underset{x}{\mathbb{R}^n} \times \underset{t}{[0,\infty)} \to \mathbb{R}$ unknown

$$\boxed{u_t + b \cdot Du = 0}$$

$D_{x,t}\, u(x,t) = (D_x u(x,t), \partial_t u(x,t))$

$D_{x,t_1} u(x,t) = (D_x u(x,t), \dots)$

$\partial_{x_1} u, \dots, \partial_{x_n} u$

$F(p, z, x) = p \cdot (b, 1) = 0$

Second order PDE.

- Laplace equation $\Delta u = 0$

where $\Delta u := \sum_{i=1}^{n} u_{x_i x_i}$, $u : \mathbb{R}^n \to \mathbb{R}$, unknown.

elliptic equations

$\Delta u = tr(D^2 u)$ $\begin{pmatrix} \partial_{x_1} u_{x_1} & \partial_{x_2} u_{x_2} & \cdots \\ \vdots & \partial_{x_2} u_{x_2} & \\ \partial_{x_n} u_{x_1} & & \partial_{x_n} u_{x_n} \end{pmatrix}$

- Heat equation $\partial_t u = \Delta u$

$u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, unknown.

parabolic equation.

- Wave equation. $\partial_{tt} u = \Delta u$

$u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$, unknown.

hyperbolic equation.

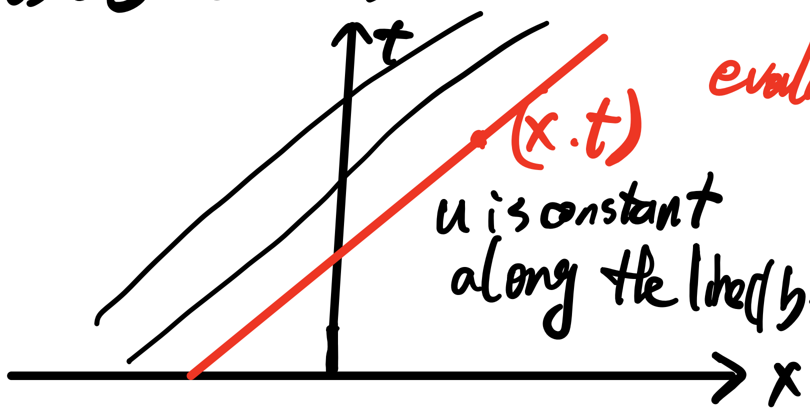
these can be used as a guide.

Transport equation.

Ex. $u_t + b\cdot Du = 0$

$u(x,t)$

$u_t(x,t) + b\cdot Du(x,t) = 0. \quad \forall (x,t)\in \mathbb{R}^n\times(0,\infty)$ (*)



evaluate $u(x,t)$ along direction $(b,1)$

suppose u smooth and solves (*)

key observation (*) $\Rightarrow$ certain partial derivatives vanishes

$$z(s) = u(x+sb,\, t+s)$$

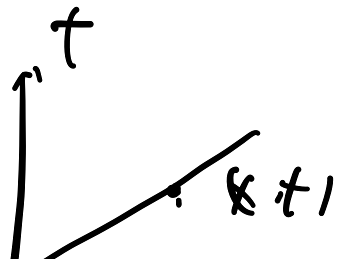
$$z'(s) = Du(x+sb,\, t+s)\,b + \partial_t u(x+sb,\, t+s) = 0$$

by transport equation

Initial value problem

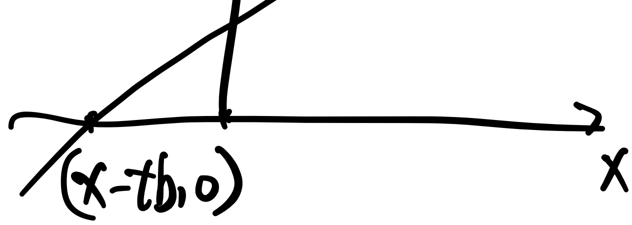
$$\begin{cases} u_t + b\cdot Du = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = g & \text{in } \mathbb{R}^n\times\{t=0\} \end{cases}$$

Solve (IVP)



$u(x+sb,\, t+s)$

let $s = -t$, (x,t)



$u(x-tb,0)=g$

Let $\boxed{u(x,t) := g(x-bt)}$

solves the transport equ.

$$\text{if } g \in C^1$$

$$u(x,t) = g(x-tb)$$

$$t=0 \rightsquigarrow u(x,0) = g(x)$$

$$\partial_t u + Du \cdot b = 0$$

$$\partial_t u = Dg(x-tb)\cdot(-b)$$

$$Du(x,t) = Dg(x-tb)$$

So $\partial_t u(x,t) + Du(x,t)\cdot b =$

$$= -Dg(x-tb)b + Dg(x-tb)\cdot b = 0$$

Is a problem <u>well-posed</u>?

- solution exist.
- unique.
- depend continuously on the data given in the problem

What properties do solutions enjoy?

e.g., regularity, finite / infinite propagation speeds

finite (wave), infinite (heat)

growth or decay

infinity.

spatial temporal.

What methods apply to analyse classes of equations?

eg, energy methods, maximum principles, Duhamel principles.
(integration by part)

Lec 10.

$U \subset \mathbb{R}^n$ open. $u: \bar{U} \to \mathbb{R}$

Hessian of u at x:

Laplacian

$$\Delta u(x) = \sum_{i=1}^{n} u_{x_i x_i}(x) = tr(D^2 u(x))$$

$\Delta u = 0$ in U. Laplace eqn. $\rightarrow$ harmonic.

$\Delta u = f$ in U Poisson eqn.

$(f: U \to \mathbb{R}$ given$)$

1D:

$u'' = 0$.

$\Rightarrow$ Linear.

2D:

Linear ...

$u(x,y) = x^2 - y^2$

$\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = 2 - 2 = 0$

Harmonic fns. are "saddle shaped"

Let $u: \mathbb{R}^2 \to \mathbb{R}$ be harmonic

Fix $x_0 \in \mathbb{R}^2$

choose coords to diagonalize $D^2u(x_0)$

$$D^2 u(x_0) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$0 = \Delta u(x_0) = \lambda_1 + \lambda_2 \Rightarrow \lambda_1 = -\lambda_2$$

In $\mathbb{R}^n$, $0 = \Delta u(x_0) = tr\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\lambda_1 + \cdots + \lambda_n = 0$

Let's look for a radially symm. harmonic function. $u(x) = V(r)$, $r = |x| = \left|\sum_{i=1}^{n} x_i^2\right|^{\frac{1}{2}}$

function of modules.

$V: \mathbb{R}_+ \to \mathbb{R}$

To write Laplacian of u in terms of $V(r)$.

$\partial_{x_i} u(x) = \partial_{x_i} V(r(x))$

$= V'(r(x)) \cdot \partial_{x_i} r(x)$ for $\partial_{x_i} r(x) = \frac{1}{2} \frac{1}{r(x)} 2x_i$

$= V'(r(x)) \frac{x_i}{r(x)}$ $= \frac{x_i}{r(x)}$

$\partial_{x_i x_i} u(x) = V''(r) \frac{x_i^2}{r^2} + V'(r) \left\{ \frac{1}{r} - \frac{x_i}{r^2} \cdot \frac{x_i}{r} \right\}$

$\Delta u(x) = \sum_{i=1}^{n} \partial_{x_i x_i} u(x)$

$= V''(r) + V'(r) \frac{n-1}{r}$

Want to find sol'n to $V''(r) + \frac{n-1}{r} V'(r) = 0$ on $(0, +\infty)$

Case 1. $V'(r) = 0$ for some r_0

$\Rightarrow$: $V' \equiv 0$ use existence & uniqueness of ODEs.

Then V constant so u constant.

Case 2. $V'(r) \neq 0$, $\forall r$.

$(\log V'(r))' = \frac{V''(r)}{V'(r)} = -\frac{n-1}{r} = (\log r^{1-n})'$

$\log v'(r) = \log r^{1-n} + C$

$v'(r) = C r^{1-n}$

$n \geq 3$, $\quad v'(r) = \frac{C}{2-n}\left(r^{2-n}\right)'$

$v(r) = a r^{2-n} + b$

$u(x) = v(|x|)$

$n = 2$, $\quad v'(r) = C(\log r)'$

$v(r) = a \log r + b$

Def. $\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)} r^{2-n}, & n \geq 3 \end{cases}$

$\alpha(n)$ = column of unit ball in $\mathbb{R}^n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}$

conti. of $n!$

$\Delta \Phi = 0$ on $\mathbb{R}^n \setminus \{0\}$

For $x \neq 0$, $\Delta \Phi(x) = 0$

For $x \neq y$ $\quad \Delta_x \Phi(x-y) = 0$

For any $f \in C_c^2(\mathbb{R}^n)$

$f(y) \cdot \Delta_x \Phi(x-y) = 0$

$\Delta_x \int \underbrace{f(y)\Phi(x-y)}_{=0,\ \forall y} dy = 0$? NO

★ $u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy$ is not harmonic

Thm. Let $f \in C_c^2(\mathbb{R}^n)$, u as in ☆

Then $\begin{cases} u \in C^2(\mathbb{R}^n) \\ -\Delta u(x) = f(x) \text{ on } \mathbb{R}^n \end{cases}$

☆ (u solves Poisson eqn)

$f \in C_c^2(\mathbb{R}^n)$ means $f \in C^2(\mathbb{R}^n)$ and $\exists K \subset \mathbb{R}^n$ compact. s.t. $f(x) = 0$ $\forall x \in \mathbb{R}^n \setminus K$

$\Delta_x \Phi(x-y) = -\delta_y(x)$ why?

$\Delta u(x) \overset{\text{formal}}{=} -\int f(y)\delta_y(x)$

$\doteq -f(x)$

Lec 11.

Recall.

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|), & n=2 \\ \frac{1}{n(n-2)\alpha(n)}|x|^{2-n} & n\geq 3 \end{cases}$$

" not exist on 0 "

Thm. Let $f\in C_c^2(\mathbb{R}^n)$ and $u(x)=\int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy$

Then. (a) $u\in C^2(\mathbb{R}^n)$

(b) $-\Delta u = f$ on $\mathbb{R}^n$

Pf: ① $u(x)$ well-defined. i.e.

$$\underbrace{\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B(x,\varepsilon)} f(y)\Phi(x-y)\,dy}_{u_\varepsilon(x)} \text{ exists}$$

$y\propto r^n$

$dy = r^{n-1}dr$

For $0<\varepsilon_1<\varepsilon_2$,

$$|u_{\varepsilon_1}(x)-u_{\varepsilon_2}(x)| = C_n\int_{B(x,\varepsilon_2)-B(x,\varepsilon_1)} f(y)\,(x-y)^{2-n}$$

$d(2\pi r)^{2-n}$

$= r^{1-n}\,dr.$

ε_1 ε_2

(if $n\geq 3$)

$$\leq C_n|f|_{C^0}\cdot\int_{B(0,\varepsilon_2)\setminus B(0,\varepsilon_1)}|y|^{2-n}\,dy$$

"sup norm"

$\int_{\varepsilon_1}^{\varepsilon_2}\int_{\partial B_r}|y|^{2-n}\,ds\,dr$

$=\int_{\varepsilon_1}^{\varepsilon_2}|y|^{2-n}r^{n-1}dr$

$\max\{f(x), x\in\mathbb{R}^n\}$

$$= C_n|f|_{C^0}\int_{\varepsilon_1}^{\varepsilon_2} r^{2-n}r^{n-1}\,dr$$

$$\leq C_n|f|_{C^0}\,\frac{\varepsilon_2^2}{2}$$

$\int_0^R r\,dr$

$u_\varepsilon(x)$ Cauchy thus converges. $u(x)$ well-defin...

Check u_ε is continuous

① $u_\varepsilon(x) = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} f(x-y)\,\Phi(y)\,dy.$

local integrability of Φ

$|u(x)-u(y)|$

$= \left|\int_{\mathbb{R}^n} \{f(x-y) - f(z-y)\}\,\Phi(y)\,dy\right|$

$|u(x)-u(y)|$

$\leq \varepsilon \cdot \int_{\tilde{B_R}} \Phi(y)\,dy$

$\leq C_{\tilde{R},n}\cdot\varepsilon$

Fix $\varepsilon>0$, choose $\delta>0$ small enough so $\sup_{y\in\mathbb{R}^n} |f(x-y)-f(z-y)| < \varepsilon$ for $|x-z|<\delta$

f cont. on compact set thus uniform cont.

② $u\in C^1$ with

$\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \Phi(y)\,\partial_{x_i} f(x-y)\,dy$

$Q(h) = \dfrac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y)\left[\dfrac{f(x+he_i-y)-f(x-y)}{h}\right]dy$

$\left|Q(h) - \int_{\mathbb{R}^n} \Phi(y)\,\partial_{x_i} f(x-y)\,dy\right|$

$= \left|\int_{\mathbb{R}^n} \Phi(y)\left\{\dfrac{f(x+he_i-y)-f(x-y)}{h} - \partial_{x_i} f(x-y)\right\}dy\right|$

$(z=x-y)$

$\leq \sup_{z\in\overline{B_R(x)}} \left|\dfrac{f(z+he_i)-f(z)}{h} - \partial_{x_i} f(z)\right| \cdot \underbrace{\int_{\overline{B_R(x)}} \Phi(y)\,dy}_{\leq C_{R,n}}$

$$= \sup_{z \in B_R(x)} \frac{1}{h} \int_0^1 Df(\gamma^z(t))\cancel{h} e_i \, dt - \underset{Df(\gamma^z(0)) e_i \, dt}{\partial_{x_i} f(z)\, dt}$$

$\to 0$ as $h \to 0$

$\gamma^z(t) = z + t \cdot h e_i$

$z \longrightarrow z + he_i$

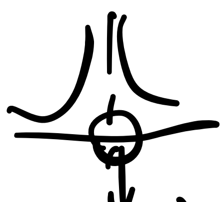
$\gamma'^z(t) = he_i$

③. $u \in C^2$ with

$$\partial_{x_i x_j} u(x) = \int_{\mathbb{R}^n} \Phi(y)\, \partial_{x_i x_j} f(x-y)\, dy$$

So $\Delta u(x) = \int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy$

$$= \int_{\mathbb{R}^n} \Phi(y)\, \underset{\partial_{yy}\ (-1)^2 = 1}{\Delta_y} f(x-y)\, dy$$



$B(0,\varepsilon)$

$$= \underbrace{\int_{B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y)\, dy}_{I(\varepsilon)} + \underbrace{\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y)\, dy}_{II(\varepsilon)}$$

$=$

$$|I(\varepsilon)| \le n \|f\|_{C^2(\mathbb{R}^n)} \int_{B(0,\varepsilon)} \Phi(y)\, dy$$

$$\le n \|f\|_{C^2(\mathbb{R}^n)} \int_0^\varepsilon r\, dr$$

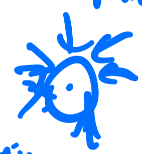
$\to 0$ as $\varepsilon \to 0$

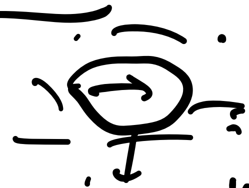
integral by part ?

$$I(\varepsilon) = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} -\nabla\Phi(y)\cdot\nabla_y f(x-y)\,dy + \underbrace{\int_{\partial B(0,\varepsilon)} \Phi(y)\,\nabla_y f(x-y)\cdot\nu_{\mathbb{R}^n\setminus B(0,\varepsilon)}}_{b_1(\varepsilon)}$$

$\Phi(y)\,|\nabla_y f(x-y)|$ outer

outward to boundary





inner unit normal.

if $\Phi \sim |x|^{2-n}$

$$\int_{\partial B_\varepsilon} \Phi = \varepsilon^{2-n}\cdot\varepsilon^{n-1} = \varepsilon$$

$$\int_0^{\varepsilon} \frac{|y|^{2-n}\,(n(n-1)\alpha(n))\,y^{n-2}}{n(n-2)\alpha(n)}\,dy = C\cdot\varepsilon$$

$$|b_1(\varepsilon)| \le |f|_{C^1(\mathbb{R}^n)} \int_{\partial B(0,\varepsilon)} \Phi(y) = C_n |f|_{C^1(\mathbb{R}^n)} \underbrace{\int_{\partial B(0,\varepsilon)} \varepsilon^{2-n}}_{\varepsilon}$$

if Φ $\log|x|$ ok.

$$= b_1(\varepsilon) + \int_{\mathbb{R}^n} \underbrace{\operatorname{div}(\nabla\Phi)}_{\Delta\Phi = 0} f(x-y)\,dy \;\underbrace{-\int_{\partial B(0,\varepsilon)} f(x-y)\,\nabla\Phi(y)\,\nu_{\mathbb{R}^n\setminus B(0,\varepsilon)}\,dy}_{b_2(\varepsilon)}$$

$\partial(\mathbb{R}^n - B(0,\varepsilon))$

$$\Delta u(x) = \lim_{\varepsilon\to 0} b_2(\varepsilon)$$

$$\nabla\Phi(y) = C_n(2-n)|y|^{-n}y \qquad \Phi \sim |x|^{2-n}$$

$$-\nabla\Phi(y)\frac{y}{|y|} = C_n(n-2)|y|^{-n+1} \qquad |\nabla\Phi| \sim |x|^{1-n}$$

$$I_2(\varepsilon) = -C_n \int_{\partial B(x,\varepsilon)} f(x-y)\, \varepsilon^{1-n}$$

$$= -C_n \int_{\partial B(0,\varepsilon)} f(x-y)\, dy$$

$$= -C_n f(x) \quad \text{choose } \Phi \text{ s.t. } C_n = 1$$

Lec 12

Def. $\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha(n)}|x|^{2-n} & n\geq 3 \end{cases}$ dimention $=2$.

$\alpha(n)$ = volumn of $B(0,1)\subset\mathbb{R}^n$

$=(n-1)$ dim surface area of $\partial B(0,1)\in\mathbb{R}^n$

$f(r) = |B(0,r)| = \alpha(n)r^n$

$f'(r) = |\partial B(0,r)| = n\alpha(n)r^{n-1}\Big|_{r=1} = \boxed{n}\alpha(n)$

$\frac{n}{r}\alpha(n)r^n = \frac{n}{r}f(r)$

if $r=1$. $f(1)=\alpha(n) = \frac{1}{n}f'(1)$

Thm. Let $f\in C_c^2(\mathbb{R}^n)$, $u(x)=\int_{\mathbb{R}^n}\Phi(x-y)f(y)dy$

Then $u\in C^2(\mathbb{R}^n)$ and

$$-\Delta u(x) = f(x)$$

$\int_{\partial B(0,\varepsilon)} \frac{|x|^{2-n}}{n(n-2)\alpha(n)}dx = \int_0^\varepsilon \frac{|x|^{2-n}}{n(n-2)\alpha(n)}\cdot n\alpha(n)r^{n-1}dr$

$=\frac{1}{2(n-2)}\varepsilon^2$

Proof. we have showed.

$$\Delta u(x) = \int_{\mathbb{R}^n}\Phi(x-y)\Delta f(y)dy$$

integral by part $\times 2$

$$=\lim_{\varepsilon\to 0} b_2(\varepsilon)$$

$b_2(\varepsilon) = -\int_{\partial B(0,\varepsilon)} f(x-y)\nabla\Phi(y)\cdot\frac{-y}{|y|}dy$ $\to$ inner unit norm. ?

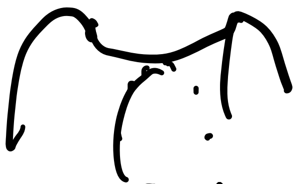
$n\geq 3$. $\nabla\Phi_y = \frac{2-n}{n(n-2)\alpha(n)}|y|^{1-n}\frac{y}{|y|}$

$\nabla\Phi_y\cdot\frac{-y}{|y|} = \frac{1}{n\alpha(n)}|y|^{1-n}$

$$b_2(\varepsilon) = \int_{\partial B(0,\varepsilon)} f(x-y)\, dS(y)$$

$$\wedge \quad \frac{-1}{n\alpha(n)\varepsilon^{n-1}}$$

$$= \frac{-1}{|\partial B(0,\varepsilon)|} \int_{\partial B(0,\varepsilon)} f(x-y) \overset{f \text{ continu..}}{=} -f(x) \quad //.$$



Harmonic

Thm. (Mean Value Formulas).

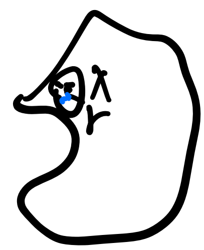
Let $u \in C^2(U)$ be harmonic ($\Delta u = 0$ in U) [open $\subseteq \mathbb{R}^n$]

only holds for harmonic

Then $\forall B(x,r) \subset U$

$$u(x) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \fint_{B(x,r)} u(y)\, dy$$

表面 体积



Proof. Let $\varphi(r) = \fint_{\partial B(x,r)} u(y)\, dS(y) = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u(y)\, dS(y)$

$z = \frac{y-x}{r}$

$dz = \frac{1}{r} dy$

$$= \frac{r^{n-1}}{n\alpha(n) r^{n-1}} \int_{\partial B(0,1)} u(x+rz)\, dS(z)$$

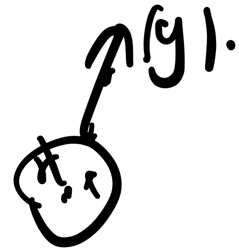
$$\varphi'(r) = \frac{d}{dr} \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz)\, dS(y)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \frac{d}{dr} u(x+rz)\, dS(z)$$

$$= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} Du(x+rz)\, z\, dS(z)$$

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} Du(y) \cdot \frac{y-x}{r}\, dS(y)$$

$\frac{y-x}{r} = \nu_{B(x,r)}(y)$



outer unit normal

why? ↓ Green's formula

$$= \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x,r)} \operatorname{div}(Du(y))\, dy = 0$$

$$\sum_{i=1}^{n} \partial_i (u_i) = \sum_{i=1}^{n} \partial_i \partial_i u = \Delta$$

Summary.

$\varphi'(r) = 0$

so $C = \varphi(r) = ⨍_{\partial B(x,r)} u(y)\, dy \overset{\text{u.c.t.n.}}{=} u(x)$ $r \to 0$.

↳ measure

$$\int_{B(x,r)} u(y)\, dy = \int_0^r \int_{\partial B(x,s)} u(y)\, dS(y)$$

$$\overset{\text{MVT}}{=} \int_0^r \left(u(x)\, n\alpha(n) s^{n-1}\right) ds$$

mean value +

$$= u(x)\, n\alpha(n) \frac{r^n}{n} = u(x) |B(x,r)|$$

divide $\Rightarrow$ $⨍_{B(x,r)} u(y)\, dy = u(x)$

Thm. Let $U \in \mathbb{R}^n$ be open bounded.

assume $u \in C^2(U) \cap C(\bar{U})$ is harmonic Then.

(i) Weak maximum principle. 边界最大 = closed U 最大

holds for general elliptic eqs

$$\max_{x \in \bar{U}} u(x) = \max_{x \in \partial U} u(x)$$

必X 边界最大 = 内部最大 = closed 最大 为 const

(ii) Strong maximum principle.

If U is connected and $\exists x_0 \in U$

s.t. $u(x_0) = \max_{x \in \bar{U}} u(x)$. then u is constant

若内点 = closed 最大 $\Leftrightarrow$ constant

Proof Note $ii \Rightarrow i$

Let $U = \bigcup_{i=1}^{\infty} U_i$, U_i is connected.

(ii). $\Rightarrow$ $\max_{x \in \bar{U}_i} u(x) = \max_{x \in \partial U_i} u(x) \le \max_{x \in \partial U} u(x)$

sup over i $\max_{\bar{U}} u(x) \le \max_{x \in \partial U} u(x) \le \max_{x \in \bar{U}} u(x)$

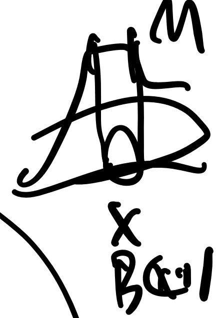
Proof. ETS (ii)

Suppose $\exists x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u(x)$

chose $r > 0$ s.t. $B(x_0, r) \subset U$. Then

$$m = u(x_0) \underset{\text{MVP}}{=} \fint_{B(x_0, r)} u(y)\,dy \le m$$

Mean-vale protety



$\Rightarrow u(y) = M$ $\forall y \in$ $B(x,r)$

Let $A = \{x \in U : u(x) = M\}$

- A is closed by continuity of u

- A is open by

why?

U is connected, so either $A = \phi$ or $A = U$

$u(x) = M$ ✓

Lec 13.

Thm. $U \subset \mathbb{R}^n$ open bounded. $u \in C^2(U) \cap C^0(\bar{U})$ is harmonic. Then

(i) Weak maximum ppl. $\max_{x \in \bar{U}} u(x) = \max_{x \in \partial U} u(x)$

(ii) (Strong maximum ppl.)

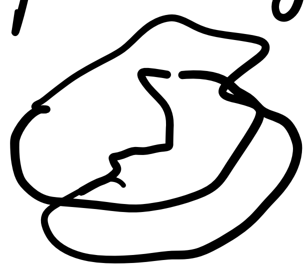
If U is connected and there is a point $x_0 \in U$. st. $u(x_0) = \max_{u \in \bar{U}} u(x)$ then u is constant

Application 1.

Dirichlet problem.

$$\begin{cases} -\Delta u = 0 & \text{in } U \in \mathbb{R}^n \text{ open boundary.} \\ u = g & \text{on } \partial U \end{cases}$$

given $g \in C(\partial U)$



Bound value problem (BVP) for Poisson equation $-\Delta = f$ in U

$$\begin{cases} -\Delta u = f & \text{in } U \quad (P) \\ u = g & \text{on } \partial U \quad (*) \end{cases}$$

if $n \ge 2$). infinite. solution to (1)

Thm (Uniqueness for BVP for Poisson equation)

Let $g \in C(\partial U)$

$U \subset \mathbb{R}^n$ open, bounded, $f \in C(U)$. Then, there is at most one solution $u \in C^2(U) \cap C^0(\bar{U})$

$$(*) \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

Proof Let u_1, u_2 be solutions to $*$.

Let $w = u_1 - u_2$, Then w solves

$$\begin{cases} -\Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases}$$

On each connected component of U, apply (strong) max principle.

weak is OK

$w \leq 0$ in $\bar{U}$ with strictly $<$ unless $w \equiv 0$

apply min. principle (max principle to $-w$)

$-w \leq 0$ with $<$ unless $w \equiv 0$

$\Rightarrow w \equiv 0$

Alt proof of weak maximum principle

(for multiple diff equ.)

(Mean Value property cannot generalize)

to show

$\Delta u = 0$ in $U \Rightarrow \max_{\bar{U}} u = \max_{\partial U} u$

step 1. If $\Delta v > 0$, the v cannot have an interior maximum in U (strong maxim.)

If it did, say at $x_0 \in U$.

$Dv(x_0) = 0$
$D^2 v(x_0) \leq 0$
NSD

$\Rightarrow \Delta v(x_0) \leq 0$

trace $(D^2 v(x_0)) \leq 0$
by diagonalize

$\Rightarrow \Leftarrow$

Step 2. Let $v^{\varepsilon}(x) = u(x) + \varepsilon e^{x_1}$

Then $\Delta v^{\varepsilon} = 0 + \varepsilon e^{x_1} > 0$ (apply step 1)

So $\max_{x \in \bar{U}} v^{\varepsilon}(x) = \max_{x \in \partial U} v^{\varepsilon}(x)$

pass $\varepsilon \to 0$, $v^{\varepsilon} \to u$ uniformly so

$\max_{x \in \bar{U}} u(x) = \max_{x \in \partial U} u(x)$

(Regularity)

Thm. $U \subset \mathbb{R}^n$ open bounded. If $u \in C(U)$ satisfies the mean value property. then $u \in C^\infty(U)$

very strong

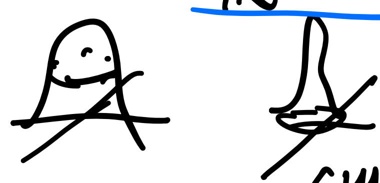
Proof. Let $\eta = \begin{cases} c \exp\left(\frac{1}{r^2-1}\right) & \text{if } r<1 \\ 0 & \text{if } r \geq 1 \end{cases}$

$r \to 1^-$, $\to -\infty$

w/ c such that

$$\int_{\mathbb{R}^n} \eta(|x|)\,dx = 1$$

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\left(\frac{|x|}{\varepsilon}\right)$$

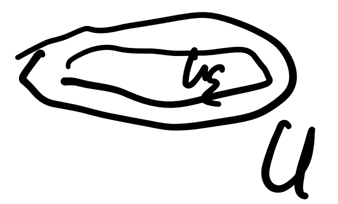


$\int_{\mathbb{R}^n} \eta_\varepsilon\,dx = 1$

support of $\eta_\varepsilon \subset B(0,\varepsilon)$

Let $U_\varepsilon = \{x \in U \ \operatorname{dist}(x, \partial U) > \varepsilon\}$

Def $u_\varepsilon : U_\varepsilon \to \mathbb{R}$ by



$$u_\varepsilon(x) = u * \eta_\varepsilon(x) = \int_{\mathbb{R}^n} u(y)\,\eta_\varepsilon(x-y)\,dy$$

Key property of convolutions.

Since $\eta_\varepsilon \in C^\infty$, $u_\varepsilon \in C^\infty(U_\varepsilon)$

Claim $u_\varepsilon(x) = u(x)$ in U_ε

(thus, $u \in C^\infty(U_\varepsilon)$ $\forall \varepsilon > 0$, thus $u \in C^\infty(U)$

Proof of claim. Fix $x \in U^{\varepsilon}$

$$u_\varepsilon(x) = \int_U \eta_\varepsilon(x-y)\, u(y)\,dy$$

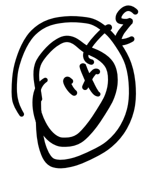
$\mathbb{R}^n \leftarrow$ outside U vanish.

$$= \frac{1}{\varepsilon^n}\int \eta\left(\frac{|x-y|}{\varepsilon}\right) u(y)\,dy$$

$B(x,\varepsilon) \rightarrow$ η vanish outside $B(x,\varepsilon)$.

$$= \frac{1}{\varepsilon^n}\int_0^{\varepsilon}\int_{\partial B(x,r)} \eta\left(\frac{\overset{r}{|x-y|}}{\varepsilon}\right) u(y)\,dS(y)\,dr$$

$$= \frac{1}{\varepsilon^n}\int_0^{\varepsilon} \eta\left(\frac{r}{\varepsilon}\right)\left(\int_{\partial B(x,r)} u(y)\,dS(y)\right)dr$$



$$= \frac{1}{\varepsilon^n}\int_0^{\varepsilon} \eta\left(\frac{r}{\varepsilon}\right)\cdot u(x)\, n\,\alpha(n)\, r^{n-1}\,dr$$

change of variable $= u(x)\cdot\int_{\mathbb{R}} \eta_\varepsilon(x)\,dx = u(x)$ //

Cor. $u \in C^2$, u harmonic $\Rightarrow u \in C^\infty$

Thm (Converse to Mean Value Theorem)

Suppose $u \in C(U)$ satisfies Mean Value Property

$$u(x) = \fint_{\partial B(x,r)} u(y)\,dS(y), \quad \forall\, B(x,r) \subset U$$

(*).

$$= \fint_{B(x,r)} u(y)\,dy$$

Then $u \in C^2$, and $\Delta u = 0$

Proof: By previous theorem, $u \in C^\infty$

Suppose u not harmonic so $\exists x_0 \in U$ s.t. $\Delta u(x_0) > 0$ (WLOG).

Since $u \in C^2$, $\exists r > 0$ s.t. $\Delta u(x) > 0$ in $B(x_0, r)$. Taking $\varphi(r) = \fint_{\partial B(x_0, r)} u(y)\, dS(y)$ ☆

we showed

$$\varphi'(r) = \frac{1}{n\alpha(n) r^{n-1}} \int_{B(x_0, r)} \Delta u(y)\, dy > 0$$

By $\varphi'(r) = 0$ by (☆).

Ex.

$\fint_{\partial B(x, r)} u\, dS \equiv \text{const}$

$\Leftrightarrow \fint_{B(x, r)} u\, dy \equiv \text{const}.$

Lect 14.

Notation

- multi-index $\alpha = (\alpha_1 \cdots \alpha_n)$ $\alpha_i \in \mathbb{N} \cup \{0\}$
- order of α is $|\alpha| = \alpha_1 + \cdots + \alpha_n$
- $D^\alpha u(x) = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$

e.g. $n=2$.

$$|D^k u(x)| = \Big(\sum_{|\alpha|=k} |D^\alpha u(x)|^2\Big)^{\frac{1}{2}}$$

Eg: $\alpha = (1,2)$

$\Rightarrow D^\alpha u(x)$

$|\alpha| = 3$ $= \dfrac{\partial^3 u(x)}{\partial x_1 \partial x_2^2}$

$n=2$

$|D^2 u(x)|$

$= \left((\partial_{11} u)^2 + (\partial_{12} u)^2 + (\partial_{21} u)^2 + (\partial_{22} u)^2 \right)^{\frac{1}{2}}$

Thm. Assume u is harmonic in $U \subset \mathbb{R}^n$ open.

Then $\forall B(x_0, r) \subset U$ for every multi index α

$$|D^\alpha u(x_0)| \le \frac{C_{k,n}}{r^{n+k}} \int_{B(x_0,r)} |u| \, dx$$

, where $k = |\alpha|$

$C_0 = \dfrac{1}{\alpha(n)}$

$C_k = \dfrac{(2^{n+1} n k)^k}{\alpha(n)}$

Proof. $k=0$

$$|u(x_0)| = \Big| \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} u \Big| \le \frac{1}{\alpha_n r^n} \int_{B(x_0,r)} |u|$$

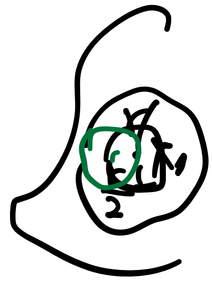
MVT

$k=1$, WLOG

$\alpha = (1, 0, \cdots 0)$. Since $u \in C^\infty$

$0 = \partial_1 \Delta u = \Delta \partial_1 u$ ↳ harmonic, can commute.

$$\partial_1 u(x_0) = \fint_{B(x_0,\frac{r}{2})} u_{x_1}\, dx \quad \text{(MVT)}.$$



$$= \frac{2^n}{\alpha(n) r^n} \int_{B(x_0,\frac{r}{2})} \nabla u \cdot e_1 \, dx$$

$$= \frac{2^n}{\alpha(n) r^n} \int_{\partial B(x_0,\frac{r}{2})} u\, \underbrace{e_1 \cdot \nu}_{\leq 1}$$

$\int_B$ Dude $= \int_{\partial B}$ (integral by part).

$$|\partial_1 u(x_0)| \leq \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))} \cdot \frac{2^n}{\alpha(n) r^n} \cdot n\alpha(n) \left(\frac{r}{2}\right)^{n-1}$$

$$= \|u\|_{L^\infty(\partial B(x_0,\frac{r}{2}))} \frac{2n}{r}$$

For each $x \in \partial B(x_0, \frac{r}{2})$, $B(x, \frac{r}{2}) \subset B(x_0, r) \subset U$

So $u(x) = \frac{2^n}{\alpha(n) r^n} \int_{B(x,\frac{r}{2})} u(y)\,dy \leq \frac{2^n}{\alpha(n) r^n} \int_{B(x_0,r)} |u|\,dy$

$$|\partial_1 u(x_0)| \leq \frac{2n}{r} \cdot \frac{2^n}{\alpha(n) r^n} \int_{B(x_0,r)} |u|\,dy$$

For $k \geq 2$, sps est holds for all β, with $|\beta| \leq k-1$. Fix α a multi-index of order k

$D^\alpha u = \partial_i (D^\beta u)$ for some $i \in \{1 \dots n\}$

and β, $|\beta| = k-1$

$\frac{\partial^2 u}{\partial x_1 \partial x_2} = \partial x_1 (\partial_{x_2} u)$

Fix $B(x_0, r) \subset U$

$D^\alpha u$ is harmonic

$\Delta D^\alpha u = D^\alpha \Delta u = 0$

$$D^\alpha u(x_0) \overset{MVP}{=} \frac{k^n}{\alpha(n) r^n} \int_{B(x_0, \frac{r}{k})} \overbrace{D^\alpha u(x)}^{\nabla D^\beta u \cdot e_i} dx$$

integ. by part

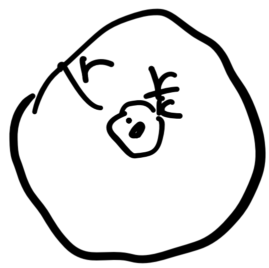
$$= \frac{k^n}{\alpha(n) r^n} \int_{\partial B(x_0, \frac{r}{k})} D^\beta u \cdot e_i \nu$$

surface area

$$|D^\alpha u(x_0)| \le \frac{k^n}{\alpha(n) r^n} \|D^\beta u\|_{L^\infty(\partial B(x_0, r/k))} \cdot \frac{n \alpha(n) k^{n-1}}{r^{n-1}}$$

$$\le \frac{kn}{r} \frac{(2^{n+1} n (n+1))^{k-1}}{\alpha(n) r^{n+k-1}} \cdot$$

can be bounded



$\int_{B(x_0, r)} |u|$

Thm (Liouville's Thm)

Suppose $u : \mathbb{R}^n \to \mathbb{R}$ harmonic and bounded

Then u is constant $|u| \le M$.

Proof. Fix $x_0 \in \mathbb{R}^n$, any r.

Derivat ests $\Rightarrow$ $|\partial_i u(x_0)| \le \frac{C_1}{r^{n+1}} \int_{B(x_0,r)} |u|$

$\le \frac{MC_1}{r^{n+1}} \alpha(n) r^n$

$= \frac{CC_1 \alpha(n)}{r}$

$\xrightarrow{r \to +\infty} 0.$

$|Du(x_0)| = 0, \forall x_0 \in \mathbb{R}^n$, mean value theorem

$\Rightarrow u \equiv \text{const.}$ (MVT. using. to prove.)

Thm Let $f \in C_c^2(\mathbb{R}^n)$, $n \ge 3$. Then any bounded solution to $-\Delta u = f$ on $\mathbb{R}^n$ is of form

$u(x) = \Phi * f(x) + C$

First we claim. $\bar{u} = \Phi * f$ is bounded

$|\bar{u}(x)| = \left| \int_{\mathbb{R}^n} \Phi(y) f(x-y)\, dy \right|$ $B(x,R)$

$\le |f|_{C^0(\mathbb{R}^n)} \int_{B(x,R)} |\Phi(y)|\, dy \xrightarrow[\text{as } |x| \to +\infty]{n \ge 3 \text{ uniformly}} ?$

$n=2$. $\Rightarrow 0$.

Now let u be any bounded soln to $-\Delta u = f$ on $\mathbb{R}^n$

$u - \bar{u} = w$ solves $\Delta w = 0$ on $\mathbb{R}^n$ and is bounded. Liouville $\Rightarrow w \equiv C$

$u = \bar{u} + C$ & C is unique..

(If $n \geq 3$, $-\Delta u = f$, u bounded $\Rightarrow u = \bar{u} + C$)

but if $-\Delta u = f$ on $\mathbb{R}^n$

$|u(x)| \to 0$ as $|x| \to +\infty$ → boundary conditions at $|x| \to +\infty$

u bounded $\quad u = \bar{u} + C$

$$\begin{cases} -\Delta u = f & U \\ u = g & \partial U \end{cases}$$

bounded by C_n

$\left(I \leq \frac{C_n}{d_x^{n-2}}\right)$

$\int_{B(0,1)} |\phi| \leq \frac{C}{d_x} \to 0$ as $d_x \to +\infty$

lec 15.

$\Delta u = 0$ in U open $\subset \mathbb{R}^n$

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} \int_{B(x_0,r)} |u|\, dx$$

for all α w/ $|\alpha| = k$, $B(x_0, r) \subset U$

$$C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1} n k)^k}{\alpha(n)}$$

For a multi-index $\alpha = (\alpha_1, \cdots \alpha_n)$, let $\alpha! = \alpha_1! \cdots \alpha_n!$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

Def. $u : U \to \mathbb{R}$ is analytic in U if $\forall x_0 \in U$. $\exists r > 0$. st $\forall x \in B(x_0, r)$

$$u(x) = \sum_\alpha \frac{1}{\alpha!} D^\alpha u(x_0) (x - x_0)^\alpha$$

Thm: If $\Delta u = 0$ in U, the u is analytic in U.

Tool 2. Multinomial thm.

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha| = k} \binom{|\alpha|}{\alpha} x^\alpha$$

$$\binom{|\alpha|}{\alpha} = \frac{|\alpha|!}{\alpha!}$$

Lec 16.

Thm (Harnack inequality)

last class (either >0 or constant)

For $U \subset \mathbb{R}^n$ open, V open connected, $V \subset\subset U$

$\exists\, C>0$ depending on V, U s.t. for any $u \geq 0$ harmonic on U.

$$\sup_V u \leq C \cdot \inf_V u$$

Proof Fix $r < \frac{1}{4}\operatorname{dist}(V, \partial U)$, last time.

$\forall\, x, y \in V$ s.t. $|x-y| \leq r$ $\quad u(x)$ & $u(y)$ close to each other

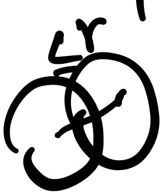
$$2^{-n}u(y) \leq u(x) \leq 2^n u(y)$$

Cover V by balls $\{B(z^i, \frac{r}{2})\}_{i=1}^N$

(Note $\bigcup_{i=1}^N B(z_i, \frac{r}{2})$ connected

by contradiction. connect V.

Fix $x, y \in V$. $x \in B_i \Rightarrow |x - z_i| < \frac{r}{2} \Rightarrow u(x) \leq 2^n u(z_i)$



$y \in B_j \Rightarrow |y - z_j| < \frac{r}{2} \Rightarrow u(y) \geq 2^{-n} u(z_j)$

can find seq of centers $\{z_{k_l}\}_{l=1}^L$ s.t.

$$|z_{k_l} - z_{k_{l+1}}| \leq r$$

$$z_{k_1} = z_i,\ z_{k_L} = z_j,\ L \leq N$$

$$u(x) \leq 2^n u(z_i) \leq \cdots \leq 2^{n(1+L)} u(z_j) \leq 2^{n(2+N)} u(y)$$

$$\Rightarrow \sup_V u \leq 2^{n(2+N)} \inf_V u$$

(N only depends on V, U)

Cor (Liouville +)

Suppose $\Delta u = 0$, on $\mathbb{R}^n$, u is bounded below (~~...~~ bounde above) or one side

Then u is constant.

Proof. $u \geq -M$ on $\mathbb{R}^n$. $(M \geq 0)$

Let $\tilde{u} = u + 2M > 0$. $(Q > 0)$

$\Delta \tilde{u} = \Delta u = 0$. harmonic.

By Harnack, fixing $x \in \mathbb{R}^n$ any $r > 0$

for all $y \in B(x, r)$

$\tilde{u}(y) \leq 2^n \tilde{u}(x)$

$\Rightarrow M \leq \tilde{u}(y) \leq 2^n \tilde{u}(x)$, $\forall y \in \mathbb{R}^n$

$\Rightarrow \tilde{u}$ bounded. harmonic, so by Liouville.

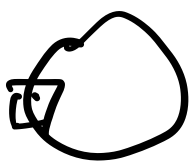
$\tilde{u} \equiv c \Rightarrow u \equiv c - M$ //

From Harnack.

Remark. We proved that if $\Delta u = 0$ on U, $|x - y| \leq r$ and $B(x, 4r)$ $B(y, 4r)$ $\subset U$ then.

$2^{-n} u(y) \leq u(x) \leq 2^n u(y)$

§8: Assume $U \subset \mathbb{R}^n$ open bounded, $\partial U \in C^1$



Thm. Let $f \in C^0(\overline{U})$, $g \in C^0(\partial U)$. There is at most one solution to

$$(*) \begin{cases} -\Delta u = f & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$$

Maximal Principle (Proved).

"Energy methods. multiply equation by a smart choice of function. integrate (by parts).

Proof. Let u_1, u_2 solves ☆ so $w = u_1 - u_2$.

$$\begin{cases} -\Delta w = 0 & \text{on } U \\ w = 0 & \text{on } \partial U \end{cases}$$

$$0 = \int_U 0 \cdot w = \int_U -\Delta w \cdot w = \int_U |\nabla w|^2 + \int_{\partial U} w \, \partial_\nu w$$

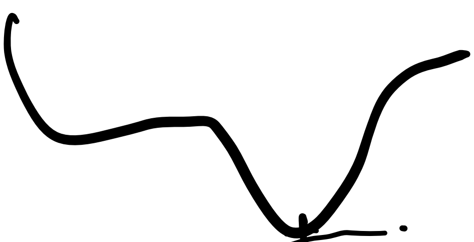
$$\int_{\partial U} w \, \partial_\nu w \to 0$$

$$= \int_U |\nabla w|^2$$

$\Rightarrow |\nabla w|^2 \equiv 0 \Rightarrow w \equiv C$ in U

$w = 0$ on ∂U $\Rightarrow w \equiv 0$ in U.

$\Rightarrow u_1 = u_2$



Euler-Lagrange.

Dirichlet Principle.

$$A = \{ w \in C^2(\bar{U}) : w = g \text{ on } \partial U \}$$

($g \in C(\partial U)$ given. $f \in C(\bar{U})$)

$$E[w] = \int_U \left[\frac{1}{2} |Dw|^2 - wf \right] dx$$

$E[w]$: energy functional

Thm: A function $u \in \mathcal{A}$ minimize $E[w]$

i.e. $\left(E[u] = \inf(\{E[w] : w \in \mathcal{A}\})\right)$

$\Longleftrightarrow$ u solves $\bigstar \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$

$E[w]$. convex.

u. $\bigstar$ is gradient ...

$\bigstar$ is the Euler-Lagrange eqn. of $E[w]$

Proof. $\Leftarrow$ Suppose u solves $\bigstar$

Choose any $w \in \mathcal{A}$ (want to show $E[u] \leq E[w]$)

$$0 = \int_U (-\Delta u - f)(u-w) = \int_U |\nabla u|^2 - \nabla u \cdot \nabla w - f(u-w) + \underbrace{\int_{\partial U} (u-w) \cdot \partial u}_{=0\ (u,w \in \mathcal{A},\ u=w=g)}$$

$$= \int_U |\nabla u|^2 - \nabla u \nabla w - fu + fw$$

$$\int_U |\nabla u|^2 - fu = \int_U \nabla u \cdot \nabla w - fw$$

$\bigstar$ Young's-Cauchy-Schwartz. $\nabla u \cdot \nabla v \leq |\nabla u| \cdot |\nabla v| \leq \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\nabla v|^2$

$$\leq \int_U \frac{1}{2}|\nabla u|^2 + \underbrace{\int_U \frac{1}{2}|\nabla w|^2 - fw}_{E[w]}$$

$$\Rightarrow E[u] = \int_U \frac{1}{2}|\nabla u|^2 - fu \leq E[w]$$

Proof. Fix $x_0 \subset U$, $r = \mathrm{dis}(x_0, \partial U)/4$

Set $M = \frac{1}{\alpha(n) r^n} \int_{B(x_0,2r)} |u|$

For $x \in B(x_0, r)$, we have $B(x, r) \subset U$. $\subset B(x,2r)$

so $|D^\alpha u(x)| \leq \frac{1}{\alpha(n)} \frac{(2^{n+1} n|\alpha|)^{|\alpha|}}{r^{n+|\alpha|}} \int_{B(x,r)} |u| dx$ $B(x,2r)$

$\leq M \left(\frac{2^{n+1} n|\alpha|}{r}\right)^{|\alpha|}$

$|\alpha|^{|\alpha|}$ grows fastest.

To bound $|\alpha|^{|\alpha|}$,

$e^k = \sum_{j=0}^{\infty} \frac{k^j}{j!} \overset{k \in \mathbb{N}}{\geq} \frac{k^k}{k!}$

$\Rightarrow e^{|\alpha|} |\alpha|! \geq |\alpha|^{|\alpha|}$

Multinomial thm, $n^k = (1 + \cdots + 1)^k = \sum_{|\alpha|=k} \binom{|\alpha|}{\alpha}$ $\frac{|\alpha|!}{\alpha!}$

$\geq \frac{|\alpha|!}{\alpha!}$

Also, $n^k \geq \sum_{|\alpha|=k} 1$

$\Rightarrow e^{|\alpha|} |\alpha|! \leq e^{|\alpha|} n^{|\alpha|} \alpha!$

So $\leq M \left(\frac{2^{n+1} n^2 e}{r}\right)^{|\alpha|} \alpha!$

For $x \in B(x_0, r)$, let

$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha$ $= \sum_{\alpha}$

want to

Claim For $|x-x_0| < \frac{r}{2^{n+2}n^2 e}$, $|R_N(x)| \to 0$ as $N \to +\infty$ shown

rewrite, $R_N(x)$ for some $t \in [0,1]$

By Taylor expan, $R_N(x) = \sum_{\boxed{|\alpha| \geq N}} \frac{D^\alpha u(x_0 + t(x-x_0))}{\alpha!}(x-x_0)^\alpha$

$= N$?

$$|R_N(x)| \leq \sum_{|\alpha|=N} \frac{1}{\alpha!} M \left(\frac{2^{n+1}n^2 e}{r}\right)^{|\alpha|} \alpha! \, |x-x_0|^{|\alpha|}$$

exercise $|y^\alpha| \leq |y|^{|\alpha|}$

bound

$$= M\left(\frac{2^{n+1}n^2 e}{r}|x-x_0|\right)^N \cdot \sum_{|\alpha|=N} 1 \quad \underline{\leq n^N}$$

choice of $|x-x_0|$

$$\leq M\left(\frac{1}{2n}\right)^N n^N = M\frac{1}{2^N} \to 0 \text{ as } N \to \infty //.$$

Thm. (hold for general elliptic). (Harnack inequality). Quantitative version of maximum principle.

Fix $U \subset \mathbb{R}^n$, open. For each connected open $V \subset\subset U$, $\exists\, C > 0$ depending on V only (not on u) s.t. for any nonnegative harmonic function u on U.

compactly contained $\bar{V} \subset U$ compact (subset)

$$\sup_V u \leq C \inf_V u$$

for x, y in V. $u(x) \leq C u(y) \leq C^2 u(x)$

$u(x) \leq \sup_V u \leq C \inf_V u \leq C u(y)$

Proof. Fix V, let $r = \frac{1}{4}\operatorname{dist}(V, \partial U)$

$= \inf_{x_0 \in V} \operatorname{dis}(x, \partial U)$

Take $x, y \in V$, with $|x-y| \le r$

$$u(x) \overset{MVP}{=} \fint_{B(x,2r)} u(z)\,dz = \frac{1}{\alpha(n)(2r)^n}\int_{B(x,2r)} u$$

$B(x,2r) \subset U$ $\qquad B(y,r) \subset B(x,2r)$

$$\ge \frac{1}{\alpha(n)(2r)^n}\int_{B(y,r)} u$$

$$\ge \frac{1}{2^n} \fint_{B(y,r)} u = \frac{1}{2^n} u(y)$$

To summarize: if $x, y \in V$, $|x-y| \le r$

then

$$\frac{1}{2^n} u(y) \le u(x) \le 2^n u(y)$$

Lec 17

$A = \{ w \in C^2(\bar{U}),\ w = g \text{ on } \partial U \}$

$U \subset \mathbb{R}^n$ open bounded C^1 boundary

5y Harnack

Mid term. 3 hours, 5 questions

No energy, No Green's

Thm. A function $u \in A$ solves $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$

$\Leftrightarrow$ $\min\{\int_U \{\frac{1}{2}|\nabla w|^2 - wf\}dx$ ✱

$\underbrace{\qquad}_{E[w]}$ $: w \in A\} = E[u]$

Pf: Last time: $\Rightarrow$ multiply equ for a by $(u-w)$ and integrate by part. (IBP)

$\Leftarrow$ Let's compute the Euler Lagrange equation associated to ✱

Since u minimize $E[u]$ in A, we know that for any $\varphi \in C_c^2(U)$ $\to$ only affect interior.

$E[\underbrace{u+\varepsilon\varphi}_{V_\varepsilon \in A}] \geq E[u]$

$E[u+\varepsilon\varphi] = \int_U \frac{1}{2}|\nabla u|^2 + \varepsilon \nabla u \cdot \nabla\varphi + \frac{1}{2}\varepsilon^2 |\nabla\varphi|^2 - fu - \varepsilon f\varphi \, dx$

smooth to ε

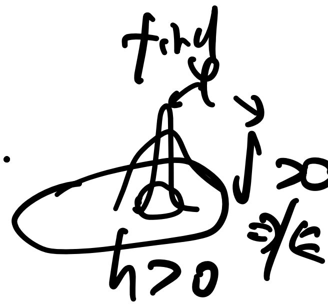
$0 \underset{\text{minimize}}{=} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E[u+\varepsilon\varphi] = \int (\nabla u \cdot \nabla\varphi - f\varphi)\, dx$

IBP. (d φ)

$= \int (-\Delta - f)\varphi\, dx + \int_{\partial U} \varphi \nabla u \cdot \nu$ $\downarrow = 0$

So for every $\varphi \in C_c^2(U)$

$\int (\underbrace{-\Delta u - f}_{h \in C(U)})\varphi = 0$

Pf. 

$\Rightarrow -\Delta - f = 0$ in U

Since $u \in A$ $\begin{cases} -\Delta = f & \text{on } U \\ u = g & \text{on } \partial U \end{cases}$

c§.

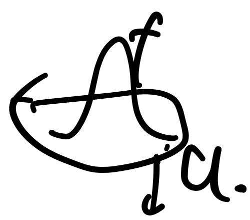
$U \subset \mathbb{R}^n$ open bounded. C^1 boundary.

Q: Can we find a representation formula for solutions to

$(*)\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$

akin to $\underline{\Phi * f \text{ on } \mathbb{R}^n}$

Φ is for $\mathbb{R}^n$ but not on U....



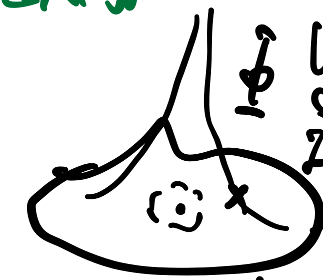
g: Bondary error...

Let's say $u \in C^2(\bar{U})$ solves $*$. Then.

$$\int_U f(y) \cancel{\Phi(x-y)}_{G(x,y)}\, dy = \int_U -\Delta u(y) \cancel{\Phi(x-y)}_{G(x,y)}\, dy$$

$$= \lim_{\varepsilon\to 0} \left[\int_V -\Delta u(y) \cancel{\Phi(x-y)}_{G(x,y)}\, dy \right]$$

Φ U: can't IBP



$V = U \setminus B_\varepsilon(x)$.

(can IBP)

$$= \lim_{\varepsilon\to 0} \left[\int_V \nabla u(y) \cdot \cancel{\nabla \Phi(x-y)}_{\nabla_y G(x,y)}\, dy - \int_{\partial V} \cancel{\Phi(x-y)}_{G(x,y)} \nabla u(y) \nu\, dS(y) \right.$$

$$= \lim_{\varepsilon\to 0} \int_V -u(y) \underbrace{\cancel{\Delta \Phi(x-y)}_{\Delta G(x,y)}}_{=0}\, dy + \int_{\partial V} u(y) \cancel{\nabla \Phi(x-y)}_{\nabla G(x,y)} \nu - \int_{\partial V} \cancel{\Phi(x-y)}_{G(x,y)} \nabla u(y) \cdot \nu$$

$\partial V = \partial U \cup \partial B_\varepsilon(x)$

So sending $\varepsilon \to 0$, $f * \Phi(x)$

$$= -\int_{\partial U} \Phi(x-y) \nabla u(y) \cdot \nu$$

$\int_{\partial B(x,\varepsilon)} u(y) \nabla \Phi(x-y) \nu$, $\varepsilon \to 0$ $\to u(x)$ contains

$$+ u(x) \quad (\text{by HW. Prob. 1})$$

$$+ \int_{\partial U} u(y) \nabla \Phi(x-y) \cdot \nu$$

$$u(x) = f * \Phi(x) - \int_{\partial U} g(y) \nabla \Phi(x-y) \cdot \nu + \int_{\partial U} \Phi(x-y) \underbrace{\nabla u(y)}_{\text{not good.}} \cdot \nu$$

want Φ for generally

to vanish on ∂U

formular to BVP given f, g

$x \in U$ fixed.

Let Φ^x solve $\begin{cases} \Delta \Phi^x = 0 \\ \Phi^x(y) = \Phi(x-y) \end{cases}$

Let $G(x,y) = \Phi(x-y) - \Phi^x$

Now replace Φ with G.

$\varepsilon \to 0$, $\int_U f(y) G(x,y)\, dy = \int_{\partial U} g(y) \nabla_y G(x,y) \cdot \nu$

$+ \lim_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} u(y) \nabla_y G(x,y) \cdot \nu \quad = u(x)$

$+ \underbrace{\int_{\partial U} G(x,y) \nabla u \cdot \nu}_{=0}$ By def of G

$+ \int_{\varepsilon \to 0} \int_{\partial B(x,\varepsilon)} G(x,y) \cdot \nabla u \cdot \nu \quad = 0$

$\Rightarrow$ Exist & uniqueness sol'n exist.

$$u(x) = \int_U f(y) G(x,y)\, dy - \int_{\partial U} g(y) \nabla_y G(x,y) \nu$$

Lec 18.

$U \subset \mathbb{R}^n$ open bounded $\partial U \in C^1$

α. can we find representation formula for solutions for

$$★ \begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

We showed if $u \in C^2(U) \cap C^1(\bar{U})$ solve ★

$$u(x) = \int_U G(x,y) f(y)\,dy - \int_{\partial U} g(y) \nabla_y G(x,y) \nu \, dS(y)$$

where $G(x,y) = \Phi(x-y) - \phi^*(y)$

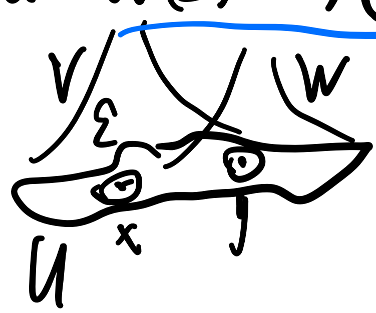
Greens function

$$\begin{cases} \Delta \phi^*(y) = 0 & \text{in } U \\ \phi^*(y) = \Phi(x-y) & \text{on } \partial U \end{cases}$$

Thm $G(x,y) = G(y,x)$ if $x \neq y$

Pf. Let $v(z) = G(x,z)$ and $w(z) = G(y,z)$

want to show $v(y) = w(x)$



$V_\varepsilon = U \setminus (B(x,\varepsilon) \cup B(y,\varepsilon))$

$\Delta v = \Delta w = 0$ on V_ε

$$0 = \int_{V_\varepsilon} \Delta v \cdot w - \Delta w \cdot v = \int_{V_\varepsilon} 0 + \int_{\partial V_\varepsilon} w \nabla v \cdot \nu - v \nabla w \nu$$

Green's identity

$$\int_{\partial V_\varepsilon} w \nabla v \nu = \int_{\partial B(x,\varepsilon)} w \nabla v \nu + \int_{\partial B(y,\varepsilon)} w \nabla v \nu + \underbrace{\int_{\partial U} w \nabla v \nu}_{0}$$

$$\int_{\partial B(x,\varepsilon)} w \nabla v \cdot \nu \to w(x) \quad \text{since } w \in C(B(x,\varepsilon_0))$$

$\nabla v \sim \varepsilon^{1-n}$, $\partial B(x,\varepsilon) \approx \varepsilon^{n-1}$

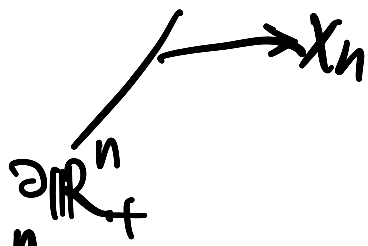
$x \Leftrightarrow \nabla v \to 1$

$$\left|\int_{\partial B(y,\varepsilon)} w \cdot \nabla v \cdot \nu\right| \leq |v|_{C^1(B(y,\varepsilon_0))} \int_{\partial B(y,\varepsilon)} |w| \to 0 \text{ as } \varepsilon \to 0.$$

$y \Leftrightarrow w$ $\to_{\varepsilon\to 0}$

$$\Rightarrow 0 = w(x) - v(y)$$

Half Space $\mathbb{R}^n_+ = \{x = (x_1 \cdots x_n) ; x_n > 0\}$



Can we explicitly find for each $x \in \mathbb{R}^n_+$,

a sol'n to $\begin{cases} -\Delta \phi^x(y) = 0 & \text{on } \mathbb{R}^n_+ \\ \phi^x(y) = \phi(x-y) & \text{on } \partial\mathbb{R}^n_+ \end{cases}$ *

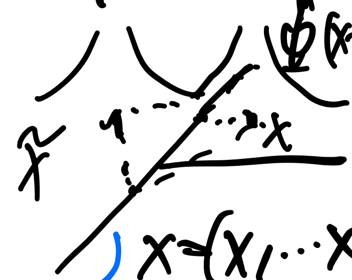
find ϕ^x

Then we'll let $G(x,y) = \Phi(x-y) - \phi^x(y)$

$n \geq 3$

$$\Phi(x-y) = \frac{|x-y|^{2-n}}{n\alpha(n)(n-2)}$$

Let $\phi^x(y) = \Phi(y - \tilde{x})$



$x = (x_1 \cdots x_n)$

$\tilde{x} = (x_1 \cdots x_{n-1}, -x_n)$

$$* \begin{cases} \Delta \phi^x(y) = 0 \text{ in } \mathbb{R}^n_+ \\ \text{Since } \forall y \in \partial\mathbb{R}^n_+, \\ |x-y| = |\tilde{x}-y|, \text{ so } \Phi(\tilde{x}-y) = \Phi(x-y) \end{cases}$$

$$G(x,y) = \frac{1}{n\alpha(n)(n-2)}\left(|x-y|^{2-n} - |\tilde{x}-y|^{2-n}\right)$$

Let's compute $-\nabla_y G(x,y)\cdot\nu$ $\quad \nu = -e_n$

$\to$ Poisson kernel.

$$= \nabla_y G(x,y)\cdot e_n$$

$$= \partial_{y_n} G(x,y)$$

$$= \frac{2-n}{n\alpha(n)(n-2)}\left(-|x-y|^{1-n}\frac{x-y}{|x-y|}e_n + |\tilde{x}-y|^{1-n}\frac{\tilde{x}-y}{|\tilde{x}-y|}\cdot e_n\right)$$

$y\in\partial\mathbb{R}^n_+ \Rightarrow |x-y| = |\tilde{x}-y|$

$$= \frac{1}{n\alpha(n)}|x-y|^{-n}\left((x-y)e_n - (\tilde{x}-y)e_n\right)$$

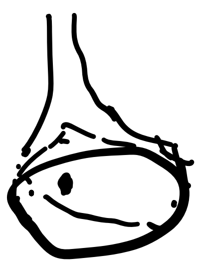
$x_n - y_n - \tilde{x}_n + y_n$

$$= \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}$$ Poisson Kernel

$B=\{x\in\mathbb{R}^n : |x|<1\}$

Can we find explicit solution to

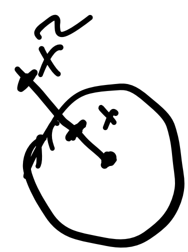
For each $x\in B$



$$\begin{cases}\Delta\phi^x(y)=0 & \text{in } B\\ \phi^x(y)=\phi(x-y) & \text{on } \partial B\end{cases}$$

For $x\in\mathbb{R}^n\setminus\{0\}$ Let $\tilde{x} = \frac{x}{|x|^2}$

$x\mapsto\tilde{x}$ called inversion through ∂B



$n \geq 3$ If $x \in B$ then $\tilde{x} \notin B$

Let $\phi^{x}(y) = \Phi(|x|(y - \tilde{x}))$

Lec 19.

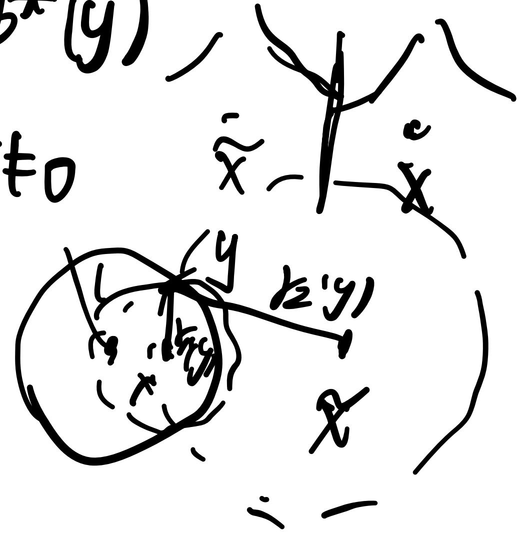
Greens fn on $B = B(0,1)$

for each $x \in B$

$$\begin{cases} -\Delta \phi^x(y) = 0 & \text{in } B \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial B \end{cases}$$

goal is to find explicit soln then

$$G(x,y) = \Phi(x-y) - \phi^x(y)$$



Inversion through ∂B : for $x \neq 0$

$$x \longmapsto \tilde{x} = \frac{x}{|x|^2}$$

Claim $\forall y \in \partial B$

$$r_1(y) = |x| \, r_2(y)$$

Proof. $r_2(y)^2 = |\tilde{x} - y|^2 = |\tilde{x}|^2 - 2\tilde{x}\cdot y + 1$

$$= \frac{|x|^2}{|x|^4} - \frac{2x\cdot y}{|x|^2} + 1$$

$$= \frac{1}{|x|^2}\left(1 - 2x\cdot y + |x|^2\right)$$

$$= \frac{1}{|x|^2}|x-y|^2$$

$$= \left(\frac{1}{|x|} r_1(y)\right)^2$$

Upshot for $y \in \partial B$

$n \geq 3$ $\quad |x|^{2-n}\Phi(y-\tilde{x}) = \Phi(|x|(y-\tilde{x}))$ since $|y-x| = |x||y-\tilde{x}|$

$= \Phi(y-x)$ and Φ is radial

Since $x \in B$ $\tilde{x} \notin B$ so.

$y = \tilde{x}$

$y \mapsto |x|^{2-n}\Phi(y-\tilde{x})$ is harmonic.

So $\underline{\Phi^x(y)} = |x|^{2-n}\Phi(y-\tilde{x})$ and $G(x,y) = \Phi(x-y) - \underline{\Phi^x(y)}$

Computing Poisson kernel

$y \in \partial B$

$x \in B$

$$-\nabla_y G(x,y)\underbrace{\nu_B(y)}_{y} = \frac{-(2-n)}{n\alpha(n)(n-2)}\left[\frac{1}{|x-y|^{n-1}}\frac{(y-x)}{|y-x|}\cdot y - |x|^{2-n}\frac{1}{|y-\tilde{x}|^{n-1}}\frac{(y-\tilde{x})}{|y-\tilde{x}|}\cdot y\right]$$

$$= \frac{1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n}$$

use $|x-y| = |x||\tilde{x}-y|$

$\tilde{x} = \frac{x}{|x|^2}$

Ex. on $B(0,r)$. $\boxed{K(x,y) = \frac{1}{n\alpha(n)}\frac{r^2-|x|^2}{|x-y|^n \cdot r}}$

Thm. Assume $g \in C(\partial B(0,r))$ and let

$$u(x) := \int_{\partial B(0,r)} g(y)K(x,y)\,dS(y)$$

Then u is smooth and $\Delta u = 0$ in $B(0,r)$

$\lim_{y \to x^0} u(x) = g(x^0) \quad \forall x^0 \in \partial B(0,r)$

Def. $\tilde{u}(x) = \begin{cases} u(x) & x \in B \\ g(x) & x \in \partial B \end{cases}$

Then $\tilde{u} \in C(\overline{B})$

Proof. For each $y \in \partial B(0,r)$, $x \mapsto K(x,y)$ is smooth

so $u(x)$ is smooth and

$$\Delta u(x) = \int_{\partial B(0,r)} g(y) \Delta_x K(x-y) dS(y) \quad ?$$

why smooth.

what is $\Delta_x K(x,y)$?

Since $\Delta_y G(x,y) = 0$ for $x \neq y$

and $G(x,y) = G(y,x)$, so $\Delta_x G(x,y) = 0$ for $x \neq y$

$$\Delta_x K(x,y) = \Delta_x(-\nabla_y G(x,y) \cdot \nu)$$

$$= \Delta_x\left(-\lim_{h \to 0}\left(\frac{G(x, y+h\nu) - G(x,y)}{h}\right)\right)$$

$$\Delta u(x) = \int_{\partial B(0,r)} g(y) \Delta_x K(x,y) dS(y) = 0$$

harmonic.

Ex. Difference quotients converge locally uniformly

Ex. If $u_h(x)$ ... of

Next class.

$u(x) \to g(x^0)$ as

$u(x) \to g$

$x \to x^0 \in \partial B$

seq of

harmonic fns w.r.t. x.

s.t. $u_k \to w$ locally uniformly

then $\Delta w = 0$

Lec 20.

Recall on $B_r(0)$

$$K(x,y) = \frac{1}{n\alpha(n)} \frac{1}{|x-y|^n} \frac{r^2-|x|^2}{r} \qquad (r=1).$$

Thm. Assume $g \in C(\partial B(0,r))$. let

$\forall x \in B_r(0)$, $u(x) := \int_{\partial B(0,r)} K(x,y) g(y)\, dS(y)$

then $u \in C^\infty(B_r(0))$ (smooth) and $\Delta u = 0$ in $B_r(0)$ ← last Lec

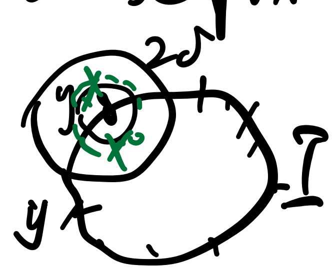
and $\forall x^0 \in \partial B_r(0)$ $\lim_{x \to x^0} u(x) = g(x^0)$

Proof of attainment of boundary data

Fix $\varepsilon > 0$. Let $\delta > 0$. st if $|y - x^0| < 2\delta$, $y \in \partial B$

then $|g(x^0) - g(y)| < \varepsilon$. Let $0 < \eta < \delta$ be fixed.

and take $x \in B$ s.t $|x - x^0| < \eta$.



we want to show

$$|u(x) - g(x^0)| \leq 2\varepsilon$$

$$|u(x) - g(x^0)| = \left| \int_{\partial B} K(x,y) g(y)\, dS(y) - g(x^0) \right|$$

Obs $\int_{\partial B} K(x,y)\, dS(y) = 1 = v(x)$ $\quad$ $v=1$, $\Delta v = 0$ dirichlet boundary.

$$= \left| \int K(x,y) [g(y) - g(x^0)]\, dS(y) \right|$$

$$\Big| \int_{\partial B} K(x,y)\,[g(y)-g(x^0)]\,dS(y) \Big|$$

$$\le \underbrace{\int_{\partial B \cap B(x^0,2\delta)} \underbrace{|g(y)-g(x^0)|}_{<\varepsilon} K(x,y)\,dS(y)}_{<\varepsilon} + \underbrace{\int_{\partial B \setminus B(x^0,2\delta)} |g(y)-g(x^0)|\,K(x,y)\,dS(y)}_{I\, <\varepsilon}$$

$$I \le 2\|g\|_{C^0(\partial B)} \int_{\partial B \setminus B(x^0,2\delta)} K(x,y)\,dS(y)$$

$$= C_n \|g\|_{C^0(\partial B)} (1-|x|^2) \int_{\partial B \setminus B(x^0,2\delta)} \frac{1}{|x-y|^n}\,dS(y)$$

Since $y<\delta$, $|x-x^0|<\delta$

for y in domain of integration $(\partial B \setminus B(x^0,2\delta))$, $|y-x^0|>2\delta$, $|y-x| = |y-x^0-(x-x^0)| \ge |y-x^0| - |x-x^0| > 2\delta-\delta = \delta$

$|x-y|>\delta$

$\Rightarrow |x-y|^{-n} < \delta^{-n}$

$$\le C_n \|g\|_{C^0(\partial B)} \delta^{-n} \underline{(1-|x|^2)} \to |x_0|^2-|x|^2 = |(x_0-x)(x^0+x)| \le |x^0-x|\,(|x^0|+|x|)$$

[illegible]

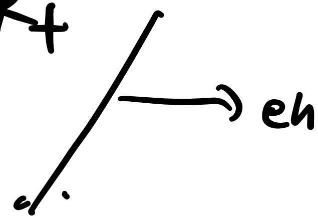
$$\le C_n \|g\|_{C^0(\partial B)} \delta^{-n} \eta < \underline{\varepsilon}$$

$I < \varepsilon$.

Thm. Let $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ → bounded

For $x \in \mathbb{R}^n_+$ let → Poisson K on $\mathbb{R}^n_+$

$$u(x) = \int_{\mathbb{R}^{n-1}} K(x,y)\, g(y)\, dS(y)$$



Then $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$

$\Delta u = 0$ on $\mathbb{R}^n_+$

and $\forall x^0 \in \partial \mathbb{R}^n_+$, $\displaystyle\lim_{\mathbb{R}^n_+ \ni x \to x^0} u(x) = g(x^0)$