Lec 21

$(\partial_t - \Delta)\, u(x,t) = 0$ Heat equation.

$x \in \underset{\text{open}}{U} \subset \mathbb{R}^n,\ t > 0$

Population dynamics

$u(x,t) =$ population (density) of $x \in \mathbb{R}^n$ at time $t > 0$

If species moves around randomly

$$u(x, t+\tau) = \fint_{\partial B_1(0)} u(x + hy, t)\, dS(y)$$

Setting $h = \tau^{\frac{1}{2}}$

$$\frac{u(x,t+\tau) - u(x,t)}{\tau} = \frac{1}{h^2} \fint_{\partial B_1(0)} [u(x+hy,t) - u(x,t)]\, dS(y)$$

Let $\tau \to 0$

LHS: $\partial_t u(x,t)$

$A(x,x) = (Ax)\cdot x$

$$\text{RHS. } \frac{1}{h^2} \fint_{\partial B_1(0)} \Big[\nabla u(x)\cdot hy + \frac{1}{2} D^2 u(x)[hy, hy]\, dS(y) + o(h^2) \Big]$$

Taylor Expansion

$$= \frac{1}{h} \nabla u(x) \cdot \underbrace{\int_{\partial B_1(0)} y\, dS(y)}_{0}$$

$$+ \frac{1}{2} \sum^{n} u_{ij}(x) \fint y^i y^j\, dS(y) + o(h)$$

$2\sum_{i,j=1} a_{ij} \int_{\partial B_1(0)} y_i y_j \, dy$

partial.

$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$ $\quad B_1(0)$

$= \frac{c_n}{2} \sum_{i=1}^{n} u_{ii}(x)$

Q. Say I have $= \tilde{c}_n \Delta u(x)$

a sol'n to heat equ.

$(\partial_t - \Delta) u = 0$

on $\mathbb{R}^n \times \{t > 0\}$

what scalings $u(\mu x, \lambda t) = u_{\mu\lambda}(x,t)$

will make $u_{\mu,\lambda}$ another sol'n to heat equ?

(if $\Delta u(x) = 0$

$u_\lambda(x) = u(\lambda x)$

$\lambda > 0$

$\Delta u_\lambda(x) = \lambda^2 (\Delta u)(\lambda x) \; \lambda > 0$

$= 0.$

$\Delta u(x) = f(x)$,

$\Delta u_\lambda(x) = \lambda^2 f(\lambda x)$)

$\partial_t u_{\mu,\lambda}(x,t) = (\partial_t u)(\mu x, \lambda t) \cdot \lambda$

$\nabla u_{\mu,\lambda}(x,t) = \nabla u(\mu x, \lambda t) \mu$

$\Delta u_{\mu,\lambda}(x,t) = \Delta u(\mu x, \lambda t) \mu^2$

$(\partial_t - \Delta) u_{\mu\lambda}(x,t) = (\lambda \partial_t u - \mu^2 \Delta u)(\mu x, \lambda t)$

Let $\lambda = \mu^2$

$\mu = \lambda^{\frac{1}{2}}$

$= \lambda(\partial_t u - \Delta u)(\mu x, \lambda t)$

Upshot. if u solves $(\partial_t - \Delta)u = 0$.

$u_\lambda(x,t) = u(\lambda^{\frac{1}{2}} x, \lambda t)$ also solves heat eqn

Let's look for sol'n to heat equ. on $\mathbb{R}^n \times \{t>0\}$

that is scaling invariance in the sense that

$$\lambda^{\alpha} u(\lambda^{\frac{1}{2}} x, \lambda t) = u(x,t) \qquad \forall \lambda > 0$$

we will search for u and α

For such a solution, taking $\lambda = \frac{1}{t}$

$$u(x,t) = \frac{1}{t^{\alpha}} u\left(\frac{x}{t^{\frac{1}{2}}}, 1\right) := \frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\frac{1}{2}}}\right)$$

where $v : \mathbb{R}^n \to \mathbb{R}$

Since u solves $(\partial_t - \Delta)u = 0$ what equ must v solve?

$$0 = (\partial_t - \Delta)u(x,t) = (\partial_t - \Delta)\left(\frac{1}{t^{\alpha}} v\left(\frac{x}{t^{\frac{1}{2}}}\right)\right)$$

$$= \frac{-\alpha}{t^{\alpha+1}} v\left(\frac{x}{t^{\frac{1}{2}}}\right) + \frac{1}{t^{\alpha}} \nabla v\left(\frac{x}{t^{\frac{1}{2}}}\right) x \left(\frac{-1}{2}\right) \frac{1}{t^{\frac{3}{2}}}$$

$$- \frac{1}{t^{\alpha}} \cdot \frac{1}{t^{\frac{1}{2}}} \cdot \frac{1}{t^{\frac{1}{2}}} \cdot \Delta v\left(\frac{x}{t^{\frac{1}{2}}}\right)$$

Let $y = \frac{x}{t^{\frac{1}{2}}}$

$$= \frac{1}{t^{\alpha+1}}\left[-\alpha\, v(y) - \frac{1}{2}\nabla v(y)\cdot y - \Delta v(y)\right]$$

To summarize

we want sol'n $u(x,t)$ to heat equation.

that is scaling-invariant

we showed such a sol'n takes form

$u(x,t) = t^{-\alpha}\, v\left(\frac{x}{t^{\frac{1}{2}}}\right)$ where

v solves

$$\alpha\, v(y) + \frac{1}{2}\nabla v(y)\cdot y + \Delta v(y) = 0$$

Lec 22.

Scale

$(\partial_t - \Delta) u(x,t) = 0 \Leftarrow \; u_\lambda(x,t) = u(\lambda^{\frac{1}{2}} x, \lambda t)$

$(\partial_t - \Delta) u_\lambda = 0 \quad \forall \lambda > 0$

We seek a sol'n to Heat Equation s.t.

$u(x,t) = \lambda^\alpha u(\lambda^{\frac{1}{2}} x, \lambda t), \; \forall \lambda > 0 \rightarrow$ scaling-invariant.

We show such a sol'n takes form

$u(x,t) = t^{-\alpha} v\left(\frac{x}{t^{\frac{1}{2}}}\right)$ for $v: \mathbb{R}^n \to \mathbb{R}$ solving

$(\star) \; \alpha v(y) + \frac{1}{2} \nabla v(y) \cdot y + \Delta v(y) = 0$

Let's seek a radially sym. sol'n to $\star$ $\;\; V(y) = w(|y|) =: w(r(y))$

$\hookrightarrow$ why? because of Δ.

$-\Delta u = f(x) = w(|x|)$

$u = f * \Phi(x) \downarrow$

w solves.

$\alpha w(r) + \frac{1}{2} w'(r) \cdot r + \frac{n-1}{r} w'(r) + w''(r) = 0$

$\partial_{e_i} v(y) = \partial_{e_i} w(r(y)) = w'(r) \partial_{e_i} r(y)$

$= w'(r) \frac{y}{|y|} e_i$

$\nabla y \cdot y = \sum \partial_{e_i} v(y) \cdot y_i = w'(r) \sum_{i=1}^{n} \frac{y_i^2}{|y|} = w'(r) \cdot r$

$\times r^{n-1}$

$0 = \underbrace{\alpha w(r) \cdot r^{n-1} + \frac{1}{2} w'(r) \cdot r^n}_{\int} + (r^{n-1} w'(r))'$

$\alpha = \frac{n}{2} \; . \; \frac{1}{2} w(r) r^n + r^{n-1} w'(r) = a$

↪ $w'(r) = -\frac{1}{2} w(r) \cdot r + \frac{a}{r^{n-1}}$

If we want v to be diff'ble at $y=0$
we need $w(0)=0$, so set $a=0$

$$\left.\begin{array}{l} (\log w)' = \frac{w'(r)}{w(r)} = -\frac{1}{2} r \\ w'(0) = 0 \end{array}\right\} \quad w(r) = b e^{-\frac{r^2}{4}}$$

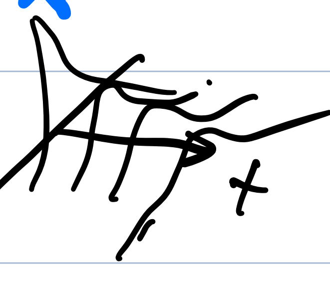
$$u(x,t) = t^{-\frac{n}{2}} \, v\left(\frac{x}{\sqrt{t}}\right)$$
$$= t^{-\frac{n}{2}} \, b \cdot e^{-\frac{x^2}{4t}}$$

Def. $\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} & t>0,\ x \in \mathbb{R}^n \\ 0 & t<0\ \ x \in \mathbb{R}^n \end{cases}$

↳ fundamental sol'n for heat equation.
aka heat kernel

- Solves heat eqn on $\mathbb{R}^n \times \{t>0\}$
- For each $t>0$, $\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1$

singular at $(0,0)$



Central limit thm. $\{x_i\}$ i.i.d. w/ $\overbrace{\text{mean}}^{\int x_i}$ μ, $\overbrace{\text{variance}}^{\int (x_i-\mu)^2}$ σ^2

$\overline{X_n} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then $\mu_{\overline{X_n}} = \mu$

$Var(\overline{X_n}) = \frac{\sigma^2}{n^2}\cdot n = \frac{\sigma^2}{n}$

CLT: $\frac{\overline{X_n}-\mu}{\ } \to N(0,1)$ $\quad (\overline{X_n} \to \mu$. LLN$)$

$\|\overline{X_n}-\mu\|_{L^2} = \sqrt{Var(\overline{X_n})} = \frac{\sigma}{\sqrt{n}}$

CLT. $\frac{\overline{X_n}-\mu}{\sigma/\sqrt{n}} \to N(0,1)$

Q: How is $\overline{X_n}$ averaging out to its mean?

A: like $\frac{\sigma}{\sqrt{n}}$ $\times\left(\frac{\sqrt{n}}{\sigma}\right)$.

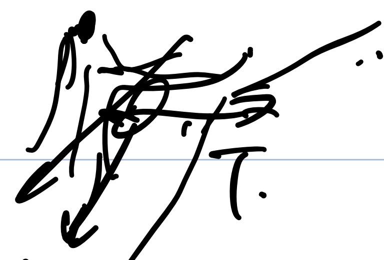
Related Q. How does heat diffuse to equilibrium?

A: like $\sqrt{t}$

Temp dist. starting from point mass at (0,0).

is $\Phi(x,t)$.

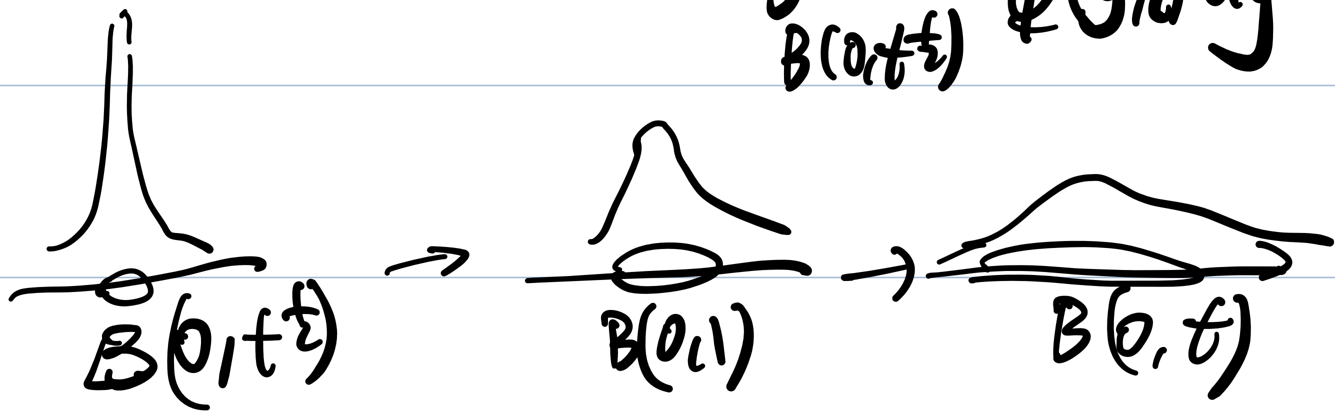
$\int_{B(0,1)} \Phi(x,1)\,dx = \frac{1}{(4\pi)^{n/2}}\int_{B(0,1)} e^{-|x|^2/4}\,dx$



$x = \frac{y}{\sqrt{t}}$, $y = x\cdot t^{\frac{1}{2}}$

$\frac{dy = dx\, t^{\frac{n}{2}}}{x \in B(0,1)} = \frac{1}{(4\pi t)^{n/2}}\int_{B(0,t^{\frac{1}{2}})} e^{-|y|^2/4t}\,dy$

$$= \int_{B(0,t^{\frac{1}{2}})} \Phi(y,t)\,dy$$



Let's.

$$\Phi(x,t) = \begin{cases} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, & t>0 \\ 0 & t=0 \end{cases}$$

- solves $(\partial_t - \Delta)\Phi(x,t) = 0$ on $\mathbb{R}^n \times (0, +\infty)$
- singular at $(0,0)$

not backward.

No short time existence.

Sps. for $g \notin C^\infty(\mathbb{R}^n)$ s.t.

$\exists t_0 > 0$. sol'n u to

$$\begin{cases} \partial_t u + \Delta u = 0 & \text{on } \mathbb{R}^n \times (0, t_0) \\ u(x,0) = g(x) \end{cases}$$

$\partial_t u + \Delta u = 0$

backward Heat Equ.

say $v(x,t)$ solves

$(\partial_t - \Delta) v = 0$

on $\mathbb{R}^n \times (0, t_0)$

$u(x,t) = v(x,-t)$

$u(x,t)$ solves.

$(\partial_t + \Delta) u = 0$

Let $v(x,t) = u(x, t_0 - t)$

solves

$$\begin{cases} (\partial_t - \Delta) v(x,t) = 0 & \text{on } \mathbb{R}^n \times (0, t_0) \\ u(x, t_0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

t_0

$\longrightarrow 0$

$\hookrightarrow g(x)$ must $\in C^\infty$ by following ..

Initial Value Problem (or Cauchy Pb.)

Given $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ find u s.t.

$$\begin{cases} (\partial_t - \Delta) u(x,t) = 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\ u(x,0) = g(x) & \text{on } \mathbb{R}^n \end{cases}$$

Thm. Assume $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ Let

$\triangle$ $u(x,t) = \int_{\mathbb{R}^n} g(y) \Phi(x-y, t)\, dy \quad x \in \mathbb{R}^n, t > 0$

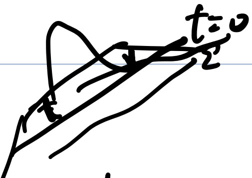
Then $u \in C^\infty(\mathbb{R}^n, (0,\infty))$

$$(\partial_t - \Delta) u(x,t) = 0, \ \forall (x,t) \in \mathbb{R}^n \times (0,+\infty)$$

$$\lim_{\substack{(x,t) \to (x_0, 0) \\ \in \mathbb{R}^n}} u(x,t) = g(x_0)$$

Cor. (Infinite speed of propagation) If $g \geq 0$, $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $g \not\equiv 0$. then $\forall (x,t) \in \mathbb{R}^n \times (0,\infty)$

$$g * \Phi > 0$$



Proof. Idea of smoothness: pass derivaties onto Φ

Fix $(x_0, t_0) \in \mathbb{R}^n \times (0, +\infty)$

$$\left| \frac{u(x+he_1,t) - u(x,t)}{h} - \int g(y) \partial_{e_1} \Phi(x-y,t) dy \right|$$

$$= \left| \int g(y) \left[\frac{\Phi(x+he_1-y,t) - \Phi(x-y,t)}{h} - \partial_e \Phi(x-y,t) dy \right] \right|$$

$$\leq \|g\|_{C^0} \int \left| \frac{\overbrace{\Phi(x+he_1-y,t) - \Phi(x-y,t)}^{} }{h} - \partial_e \Phi(x-y,t) dy \right|$$

$$= h \int_0^1 \partial_{e_1} \Phi(x+se-y,t) ds$$

$$\leq \|g\|_{C^0} \int_{\mathbb{R}^n} \int_0^1 \underbrace{|\partial_{e_1} \Phi(x+se_1-y,t) - \partial_{e_1} \Phi(x-y,t)|}_{\text{estimate this}} ds \, dy$$

Proof via smoothness proof. know Details in Evans

$$(\partial_t - \Delta)\, u(x,t) = \int_{\mathbb{R}^n} g(y)\, \underbrace{(\partial_t - \Delta)\, \Phi(x-y,t)}_{=0}\, dy$$

So u solves HE on $\mathbb{R}^n \times (0,\infty)$

Convergence to initial data :

Fix $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$

wts $\exists \delta > 0$ s.t. if

$|x - x_0| < \delta$, $|t| < \delta$, then $|u(x,t) - g(x_0)| < \varepsilon$

Choose $\tilde{\delta} > 0$ s.t. if $|x - x_0| < \tilde{\delta}$ then $|g(x) - g(x_0)| < \varepsilon$ $\to$ continuity in g

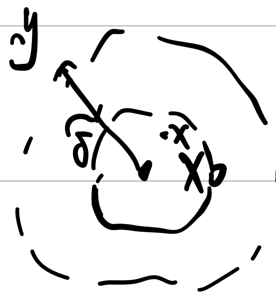
Take $|x - x_0| < \frac{\tilde{\delta}}{2}$ $\qquad$ Φ integrated to 1

$$|u(x,t) - g(x_0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)\,(g(y) - g(x_0))\, dy \right|$$

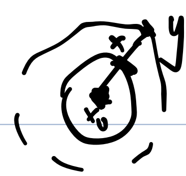
$$\le \underbrace{\int_{B(x_0,\tilde{\delta})} \Phi(x-y,t)\, \underbrace{|g(y) - g(x_0)|}_{<\varepsilon}\, dy}_{\left(\le \int_B \Phi\, dy\, \varepsilon < \int_{\mathbb{R}} \Phi\, dy\, \varepsilon < \varepsilon\right)} + \underbrace{\int_{\mathbb{R}^n \setminus B(x_0,\tilde{\delta})} \Phi(x-y,t)\, |g(y) - g(x_0)|\, dy}_{\mathrm{II} < \varepsilon\ ?}$$

check for $|t| < 1$

$$\mathrm{II} \le 2\|g\|_{C^0} \int_{\mathbb{R}^n \setminus B(x_0,\tilde{\delta})} (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}\, dy$$



$$\left(\begin{array}{l} \forall\, y \in \mathbb{R}^n \setminus B(x_0,\tilde{\delta}) \\ |x - y| \ge \frac{1}{2}|x_0 - y| \end{array} \right).$$



$$\le 2\|g\| \int (4\pi t)^{-\frac{n}{2}} e^{-\frac{|y-x_0|^2}{8t}}\, dy$$

$$\leq 4\|g\|_{C^0} \int_{\mathbb{R}^n \setminus B(x_0,\tilde{\delta})} \cdots \xrightarrow{\text{exercise}} 0 \quad \text{as } t \to 0.$$

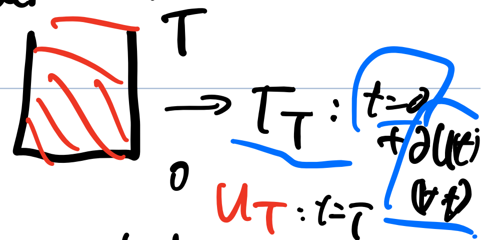
Lec 24.

$U \subset \mathbb{R}^n$ open bounded $T>0$

$$\boxed{\begin{cases} (\partial_t - \Delta) u = 0 & \text{on } U_T \\ u = g & \text{on } \Gamma_T \end{cases}}$$

$U_T = U \times (0, T]$, parabolic cylinder

$\Gamma_T = \overline{U_T} \setminus U_T$. parabolic boundary



maximum ppl tools. energy methods

Thm. (Weak max principle) Let

$u \in C_1^2(U_T)$ (C^2 in space, C_1 in time)

and $(\partial_t - \Delta) u \le 0$ (u subsolution to Heat Equation)

Then. $\max\limits_{\overline{U_T}} u = \max\limits_{\Gamma_T} u$

if supersolution $(\partial_t - \Delta) u \ge 0$ $\Rightarrow \min\limits_{U_T} u = \min\limits_{\Gamma_T} u$

Rmk Sol'ns to heat eqn satisfy a mean value property (see chap. 2 of Evans)

Proof. ① If $(\partial_t - \Delta) v < 0$, then v cannot have a maximum at $(x_0, t_0) \in U_T$: if so $D^2 v(x_0, t_0) \le 0$ and $\partial_t v(x_0, t_0) \ge 0$ (maybe)

$(\partial_t-\Delta)v(x_0,t_0)\geq 0$

contradicting $(\partial_t-\Delta)v<0$

$\hookrightarrow$ at T: ...

$\partial_t u(x_0,t_0)>0$

at $U_T\setminus T$

$\partial_t=0$

② Let $v_\varepsilon=u-\varepsilon t$

$$(\partial_t-\Delta)v_\varepsilon=\underbrace{(\partial_t-\Delta)u}_{\leq 0}-\varepsilon\leq-\varepsilon \text{ on } U_T$$

So by ①. $\max\limits_{\overline{U_T}} v_\varepsilon=\max\limits_{\Gamma_T} v_\varepsilon$

U_T bounded $\varepsilon\to 0$, $v_\varepsilon\to u$ uniformly

$$\Rightarrow \max_{\overline{U_T}} u=\max_{\Gamma_T} u$$

Cor (Comparison principle) If $g_1\leq g_2$, $f_1\leq f_2$ and $u_1,u_2\in C_1^2(U_T)\cap C(\overline{U_T})$ solve

$$\begin{cases}(\partial_t-\Delta)u_i=f_i & \text{on } U_T\\ u_i=g_i & \text{on } \Gamma_T\end{cases},\quad i=1,2.$$

if $f_1=f_2=0$, then $u_1\leq u_2$ on U_T

Proof 1. Let $w=u_2-u_1$ so $(\partial_t-\Delta)w=f_2-f_1\geq 0$

(supersol'n)

By maxim. principle. applied to $-w$.

$$\min_{U_T} w=\min_{\Gamma_T} w \underset{\text{with}}{=}=\min_{\Gamma_T}(g_2-g_1)\geq 0$$

$w = g_1$

on Γ_T

$\Rightarrow w \geq 0 \Rightarrow u_2 \geq u_1$ in U_T //.

Cor (Uniqueness). There is at most one sol'n $u \in C_1^2(U_T) \cap C(\overline{U_T})$ to

$$(*) \begin{cases} (\partial_t - \Delta) u = f & \text{on } U_T \\ u = g & \text{on } \Gamma_T \end{cases}$$

Pf let u_1, u_2 be solutions to $(*)$

Comparison principle. $\Rightarrow u_1 \leq u_2 \leq u_1$ in U_T

$\Rightarrow u_1 = u_2$ //

Proof 2. assuming ∂U is C^1 (Energy methods).

Let u_1, u_2 be sol'ns, $w = u_2 - u_1$

so w solves.
$$\begin{cases} (\partial_t - \Delta) w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Multiply by w and integral by part

For $t > 0$. $\displaystyle\int_U \partial_t w \cdot w \, dx = \int_U \Delta w \cdot w \, dx$

$\partial_t w(x,t)$

$w(x,t)$

$$= \underbrace{\int_U -|\nabla w|^2 dx}_{\leq 0} + \underbrace{\int_{\partial U} w \nabla w \cdot \nu}_{= 0 \text{ on } \Gamma_T.}$$

LHS $= \int_{\Omega} \frac{1}{2} \frac{d}{dt} w(x,t)^2 dx$

exercise

$= \frac{d}{dt}\left(\frac{1}{2}\int_{\Omega} w(x,t)^2 dx\right) \leq 0. \quad \forall t.$

i.e. $\int_{\Omega} w^2(x,t) dx$ non-increasing.

non-negative.

and $\int_{\Omega} w(x,0)^2 dx = 0$

lim $t \to 0$.

$w = g_1 - g_2 \big|_{\Gamma} = 0.$

HW. $\mathbb{R}^n \times (0,t]$

uniqueness on $\mathbb{R}^n$. only 1 calculation. $\forall t \Rightarrow w = 0$ on $\mathbb{R}^n$ & exp

$w = 0$ on $\mathbb{R}^n \times t = 0$.

Lec 25

Perspective 1. Mult. equ. by a f.n. IBP.

$$(\partial_t - \Delta) u = 0 \quad \text{on } U_T$$

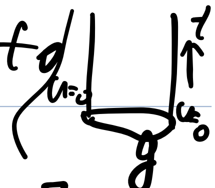
$$u = 0 \quad \text{on } \partial U \times (0,T]$$

$$\frac{d}{dt}\frac{1}{2}\int u^2 = \int_U \partial_t u\, u\, dx = \int_U \Delta u \cdot u\, dx \overset{IBP}{=} -\int_U |\nabla u|^2$$

choose an energy and differentiate along equ.

$$\underbrace{\frac{d}{dt}\frac{1}{2}\int_U u(x,t)^2 dx}_{\text{energy } e(t)} = \dots = -\int_U |\nabla u|^2$$

Thm. $U \subset \mathbb{R}^n$ open, bounded $\partial U \in C^1$, let $u \in C_1^2(U_T) \cap C^0(\bar{U}_T)$ be a solution to



Stability

$$(\star)\begin{cases} (\partial_t - \Delta) u = 0 & \text{in } U_T \\ u = 0 \ \ h(x,t) & \text{on } \partial U \times [0,T] \\ u = g & \text{on } U \times \{t = 0\} \end{cases}$$

Then $\forall t \in [0,T]$

$+2\int h(x,t)\nabla u(x,t)\cdot\nu$ $\quad \frac{\partial u}{\partial \nu}$

$$\int_U u^2(x,t)\, dx + 2\int_0^t \int_U |\nabla u(x,s)|^2\, dx\, ds \le \int_U g^2 dx$$

Ex: Apply to $\underline{u = u_1 - u_2}$ $\quad$ bounding noise/control diff.

u_1, u_2 solve $(\star)$ with diff. g.

if initial data are close, then sol'n stay close.

Pf: $\frac{d}{ds}\int_U u(x,s)^2\,dx = 2\cdot\int_U u_s(x,s)u(x,s)\,dx$ (and g)

$\partial_t u = \Delta u$

$$\overset{\text{eqn}}{=} 2\int_U \Delta u(x,s)u(x,s)\,dx$$

$$\overset{\text{IBP}}{=} -2\int |\nabla u(x,s)|^2\,dx \quad + 2\int_{\partial U} u\cdot\nabla u\cdot\nu \quad 2\int_{\partial U} h(x,t)\,\nabla u\cdot\nu$$

Integrate from 0 to t:

$$\int_U u(x,t)^2\,dx - \int_U \underbrace{u(x,0)}_{g(x)}{}^2\,dx = -2\int_0^t\int_U |\nabla u(x,s)|^2\,dx\,ds$$

Thm. Same hypothesis on U. Let $u\in C_1^2(U_T)\cap C(\bar{U}_T)$ solve

$$\begin{cases}(\partial_t-\Delta)u(x,t)=f(x) & \text{in } U_T\\ u=0 & \text{on } \Gamma_T\end{cases}$$

Then $\forall\, t\in(0,T]$

$$\int_U u^2(x,t)\,dx \le C\int_0^t\int_U |f(x,s)|^2\,dx\,ds$$

heat sources and close

for some constant C. $=C(n,U)$ not on f & u.

Proof. $\frac{d}{dt}\frac{1}{2}\int_U u(x,t)^2\,dx = \int_U \partial_t u(x,t)\,u(x,t)\,dx$

$$\overset{\text{eqn.}}{\underset{\partial_t u=\Delta u+f}{=}} \int_U \Delta u(x,t)\,u(x,t)\,dx + \int_U f(x,t)u(x,t)\,dx$$

$$\overset{\text{IBP}}{=} -\int_U |\nabla u(x,t)|^2 + \int_U \underline{f(x,t)\,u(x,t)}\,dx \quad \le \frac{1}{2\varepsilon} f(x,t)^2$$

$\le \frac{1}{2} f(x,t)^2 + \frac{1}{2} u(x,t)^2 \quad + \frac{\varepsilon}{2} u(x,t)^2$

Recall, on hw.

we solve, $\forall U \subset \mathbb{R}^n$ open-bounded, C^1, $\exists \lambda_1(U)$, s.t.

$$\forall u \in C_c^2(U),\ \lambda_1 \int u^2 \le \int |\nabla u|^2$$

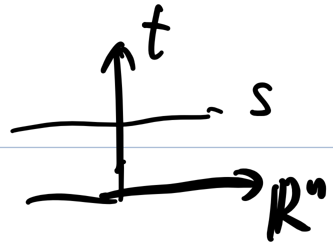
So. $\le (-\lambda_1(U)) \int_U u(x,t)^2\,dx + \frac{1}{4\lambda_1(U)} \int f(x,t)^2\,dx$ $\quad$ Let $\frac{\varepsilon}{2} = \lambda_1(U)$; $\quad \cancel{+\lambda_1(U) \int u(x,t)^2\,dx}$

$$\frac{d}{dt} \int u(x,t)^2\,dx \le \frac{1}{2\lambda_1(U)} \int_U f(x,t)^2\,dx$$

$$\Rightarrow \int_U u(x,t)^2\,dx - 0 \le \frac{1}{2\lambda_1(U)} \int_0^t \int_U f(x,s)^2\,dx\,ds \quad // \checkmark$$

Next. Question can we find sol'n formula for sol'ns to

$$(\star) \begin{cases} (\partial_t - \Delta) u = f(x,t) & \text{on } \mathbb{R}^n \times \{t>0\} \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$



u describes temp of system
w. external heat source $f(x,t)$

View this as collection of systems starting at time s w/ initial heat distribution $f(x,s)$

know Sol'n to $\left\{\begin{array}{ll}(\partial_t - \Delta)\tilde{u} = 0 & \text{on } \mathbb{R}^n \times \{t>s\} \\ \tilde{u}(x,s) = f(x,s) & \text{on } \mathbb{R}^n \{t=s\}\end{array}\right.$

$\tilde{u} = \tilde{u}(x,t;s)$

is $\tilde{u}(x,t;s) = \int \Phi(x-y, t-s) f(y,s)\, dy$

Suggests sol'n u to ☆ is given by

$$u(x,t) = \int_0^t \tilde{u}(x,t;s)\, ds$$

Duhamel's Formula

Lec 26

$$\star \begin{cases} (\partial_t - \Delta) u = f(x,t) & \text{on } \mathbb{R}^n \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases}$$

Thm Assume $f \in C_1^2(\mathbb{R}^n \times [0,+\infty))$ compactly support

Let $u(x,t) = \begin{cases} \int_0^t \underbrace{\int_{\mathbb{R}^n} \Phi(x-y;t-s) f(y,s)\,dy}_{\tilde{u}(x,t;s)}\,ds, & t>0 \\ 0 & t=0 \end{cases}$

Then $u \in C_1^2(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,+\infty))$ solves $\star$

$\tilde{u}(x,t;s)$ solves $\begin{cases} (\partial_t - \Delta)\tilde{u}(\cdot;s) = 0 & \text{on } \mathbb{R}^n \times (s,+\infty) \\ \tilde{u}(\cdot,\cdot;s) = f(\cdot,s) & \text{on } \mathbb{R}^n \end{cases}$

$t=s$

t

Proof: For each $s>0$

$$\tilde{u}(x,t;s) = \int_{\mathbb{R}^n} \Phi(y,s) f(x-y;t-s)\,dy$$

usual argument

$$\partial_t \tilde{u}(x,t;s) = \int_{\mathbb{R}^n} \Phi(y,s)\partial_t f(x-y;t-s)\,dy$$

by smooth uniform C^1 ..

$$\Delta_x \tilde{u}(x,t;s) = \int_{\mathbb{R}^n} \Phi(y,s)\Delta_x f(x-y,t-s)\,dy$$

By FTC, $u \in C_1^2(\mathbb{R}^n \times (0,\infty))$ (by $u(x,t) = \int_0^t \tilde{u}(x,t;s)\,ds$)

$$\partial_t u(x,t) = \tilde{u}(x,t;t) + \int_0^t \partial_t \tilde{u}(x,t;s)\,ds$$

$$\Delta u(x,t) = \int_0^t \Delta_x \tilde{u}(x,t;s)\,ds$$

$$(\partial_t - \Delta) u(x,t) = \underbrace{\int_{\mathbb{R}^n} \Phi(y,t) f(x-y,0)\,dy}_{\to 1} \quad (s \leftarrow t).$$

$$+\int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\left[\partial_t f(x-y,t-s) - \Delta_x f(x-y,t-s)\right] dy\,ds$$

$$= I + \underbrace{\int_0^\varepsilon \cdots}_{II} + \underbrace{\int_\varepsilon^t \cdots}_{III}$$

$$\Phi * f = \int_{\Omega_t} \Phi(y,s) f(x-y,t-s)\,dy\,ds$$

Φ is singular at $(0,0)$

$\Omega_t = \mathbb{R}^n \times (0,t)$

$\underline{(0,\infty)} \to f := 0$ if $t<0$

$$|II| \le \int_0^\varepsilon \|f\|_{C_1^2(\mathbb{R}^n)} \underbrace{\int_{\mathbb{R}^n} \Phi(y,s)\,dy}_{1} \le \varepsilon \|f\|_{C_1^2} \to 0 \text{ as } \varepsilon \to 0.$$

$$III = \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\left[\partial_t f(x-y,t-s) - \Delta_x f(x-y,t-s)\right] dy\,ds.$$

$$\partial_t f \to -\partial_s f(x-y,t-s), \quad \Delta_x f \to \Delta_y f(x-y,t-s)$$

$$= \cancel{\int_\varepsilon^t \int_{\mathbb{R}^n} \partial_s \Phi(y,s) f(x-y,t-s)}$$

$$- \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-s)\,dy \Big|_{s=\varepsilon}^{s=t}$$

$$+ \int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{\partial_s \Phi(y,s) f(x-y,t-s) - \Delta_y \Phi(y,s) f(x-y,t-s)}_{=0} \,dy\,ds$$

$$+ 0.$$

$$\cancel{= 0.}$$

$$= -I + \int_{\mathbb{R}^n} \Phi(y,s) f(x-y,t-\varepsilon)\,dy$$

$$\Rightarrow (\partial_t - \Delta) u(x,t) = I + II + III = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon) f(x-y,t-\varepsilon)\,dy$$

$= f(x,t)$ (exercise).

Finally $|u(x,t)| \le \int_0^t \left| \int_{\mathbb{R}^n} \Phi(y,s) f(x-y, t-s)\, dy \right| ds$

$\le \|f\|_{C^0} \int_0^t \underbrace{\int_{\mathbb{R}^n} \Phi(y,s)\, dy}_{=1}\, ds \le t \|f\|_{C^0}.$

So $u(\cdot, t) \to 0$ uniformly in x as $t \to 0$

So $u \in C^0(\mathbb{R}^n \times [0, \infty))$

Next: wave. $(\partial_{tt} - \partial_{xx}) u$

$\downarrow$

$$\underbrace{(\partial_t - \partial_x)}_{\text{transport } b=-1.} \underbrace{(\partial_t + \partial_x) u}_{b=1}$$

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Wave Equation. $u: U \times (0,\infty) \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open

$$\partial_{tt} u - \Delta u = 0$$

1D: initial value problem

$$\begin{cases} \partial_{tt} u - \partial_{xx} u = 0 & \text{on } \mathbb{R}\times(0,+\infty) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

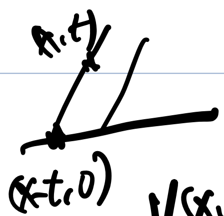
$\Downarrow$ 2-order. 2 initial conditions to uniquely determine solution.

Derive sol'n function.

$$0 = (\partial_{tt} - \partial_{xx})u = (\partial_t + \partial_x)\underbrace{(\partial_t - \partial_x)u}_{v(x,t)}$$

Then v solves transport eqn. $(b=1)$

$$\begin{cases} \partial_t v + \partial_x v = 0 \quad (b=1) & \text{on } \mathbb{R}\times(0,+\infty) \\ v(x,0) = (\partial_t u - \partial_x u)(x,0) \\ \qquad = h(x) - g'(x) := a(x) \end{cases}$$



$$v(x,t) = a(x-t)$$

$$\partial_t u(x,t) - \partial_x u(x,t) = a(x-t)$$

inhomog. transport equ. $\underline{b=-1}$. $f(x,t) = a(x-t)$ on $\mathbb{R}\times(0,+\infty)$

$$u(x,0) = g(x)$$

Last class: $u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\,ds$

sol'n for transport eqn

$$= g(x+t) + \int_0^t a(x-(s-t)-s)\,ds$$

$$= g(x+t) + \int_0^t \left[h(x+t-2s) - g'(x+t-2s)\right] ds$$

$y = x+t-2s$

$dy = -2ds$

$$= g(x+t) - \frac{1}{2}\int_{x+t}^{x-t} h(y) - g'(y)\, dy$$

$$= g(x+t) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy - \frac{1}{2}\left(g(x+t) - g(x-t)\right)$$

$$= \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$$

Summary

Thm. (D'Alembert's Formular)

Let $n=1$, $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$. Then

$$u(x,t) := \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$$

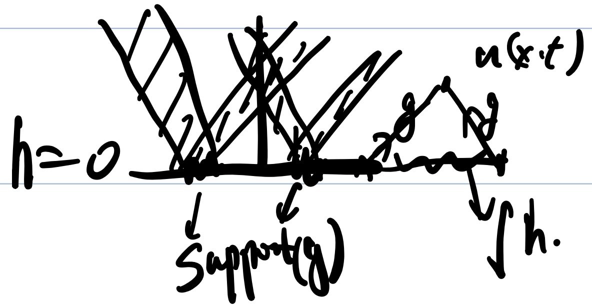
is $C^2(\mathbb{R}\times(0,\infty))$ solves $(\partial_{tt} - \partial_{xx})u(x,t) = 0$ in $\mathbb{R}\times(0,\infty)$ and $\lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0)$

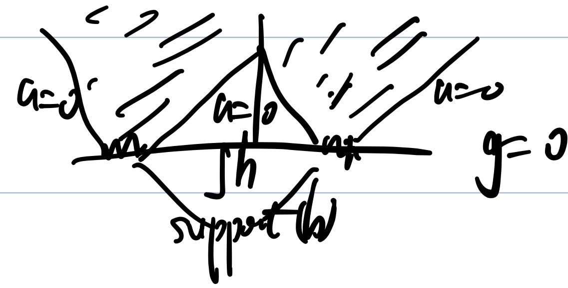
$$\lim_{(x,t)\to(x_0,0)} \partial_t(x,t) = h(x_0)$$

Rmk ① Wave equ. not regularizing (no maximum principle)

(no gain in regu. from initial data)

②. Finite speed of propogation





speed of propogation is 1

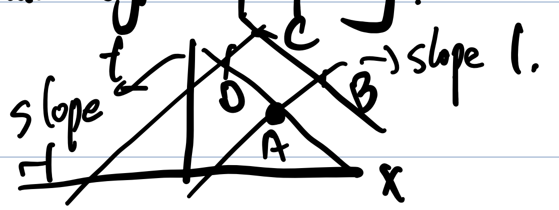
$$u(x,t) = \underbrace{\tfrac{1}{2}\left(g(x+t)\right.}\ \underbrace{\left.+\,g(x-t)\right)}\ + \tfrac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

$$= F(x+t) + G(x-t) \qquad \underbrace{\int_0^{x+t} h\,dy} + \underbrace{\int_{x-t}^{0} h\,dy}$$

③ $u(x,t) = F(x+t) + G(x-t)$

traveling wave to the left — traveling wave to the right

Parallelogram property:



Claim.

$u(A)+u(C) = u(B)+u(D)$

Pf. F const along \\ $\Rightarrow F(A)=F(D)$

$F(B)=F(C)$

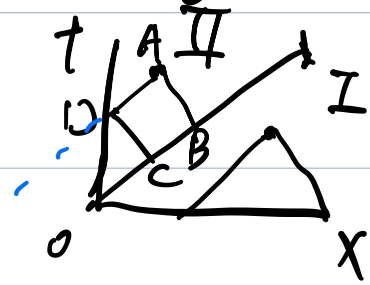
G const along // $\quad G(A)=G(B)$

$G(C)=G(D)$

$$u(A)+u(C) = F(A)+F(C)+G(A)+G(C)$$
$$= F(D)+F(B)+G(B)+G(D).$$

Ex. sol'n formula for

$$\begin{cases} (\partial_{tt}-\partial_{xx})u = 0 & \text{on } \{x>0\}\times\{t>0\} \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \\ u(0,t) = 0 \end{cases}$$



For $(x,t)\in \mathrm{I}$

$$u(x,t) = \frac{1}{2}\big(g(x+t)+g(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy$$

$$u(x,x) = \frac{1}{2}\big(g(2x)+g(0)\big) + \frac{1}{2}\int_0^{2x} h(y)\,dy$$

For $(x,t)\in \mathrm{II}$

$$u(x,t) = u(A) = \underbrace{u(D)}_{=0} + \underbrace{u(B)}_{=u(x_B,x_B)} - \underbrace{u(C)}_{=u(x_C,x_C)}$$

$$= \frac{1}{2}g(2x_B) - \frac{1}{2}g(2x_C)$$
$$+ \frac{1}{2}\int_0^{2x_B} h(y)\,dy - \frac{1}{2}\int_0^{2x_C} h(y)\,dy$$

$$u(x,t) = \frac{1}{2}\left(g(2x_B) - g(2x_C)\right) + \frac{1}{2}\int_{x_C}^{x_B} h(y)\,dy$$

$x_B = \frac{1}{2}(x+t)$ $\Leftarrow$ $B = (x+s, t-s)$

s.t. $x+s = t-s$

$x_C = \frac{1}{2}(t-x)$ $\quad s = \frac{1}{2}(t-x)$

For $(x,t) \in \mathrm{II}$

$$u(x,t) = \frac{1}{2}\left(g(x+t) - g(t-x)\right) + \frac{1}{2}\int_{t-x}^{t+x} h(y)\,dy$$

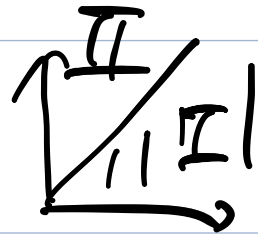
Lec 28.

$$\begin{cases}(\partial_{tt}-\partial_{xx})u=0 & \text{on } \mathbb{R}\times(0,+\infty)\\ u=g,\ u_t=h & \text{on } \mathbb{R}\times\{t=0\}\end{cases}$$

D'Alembert... $\rightarrow$

$$u(x,t)=\frac{1}{2}\big(g(x+t)+g(x-t)\big)+\frac{1}{2}\int_{x-t}^{x+t}h(y)\,dy$$

$$\begin{cases}(\partial_{tt}-\partial_{xx})u=0 & \text{on } (0,\infty)\times(0,\infty)\\ u=g,\ u_t=h & \text{on } (0,\infty)\times\{t=0\}\\ u=0 & \text{on } \{x=0\}\times(0,\infty)\end{cases}$$

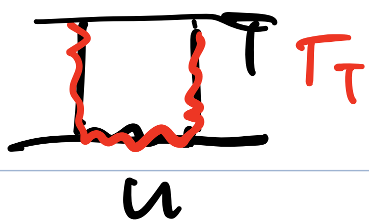


$$u(x,t)=\begin{cases}\frac{1}{2}\big(g(x+t)+g(x-t)\big)+\frac{1}{2}\int_{x-t}^{x+t}h(y)\,dy\\ \frac{1}{2}\big(g(x+t)-g(t-x)\big)+\frac{1}{2}\int_{t-x}^{t+x}h(y)\,dy\end{cases}$$

§ $U\subset\mathbb{R}^n$ open bounded $\partial U\in C^1$

$U_T=U\times(0,T]$

$\Gamma_T=\overline{U_T}\setminus U_T$



Thm. (Uniqueness) There at most one solution to

$$(*)\quad\begin{cases}(\partial_{tt}-\Delta)u=f & \text{on } U_T\\ u=g & \text{on } \Gamma_T\\ u_t=h & \text{on } U\times\{t=0\}\end{cases}$$

Pf: Let u_1, u_2 solve $(*)$, Let $w=u_1-u_2$

w solves $f=g=h=0$

$w=0$ on $\partial U\times(0,T)$ $\partial_t w=0$

energy method

$$0=\int_U(\partial_{tt}w-\underbrace{\Delta w)\cdot w_t}_{\text{IBP}}\,dx=\int_U w_{tt}w_t+\nabla w\cdot\nabla w_t+\int_{\partial U}\underbrace{\nabla w\cdot\nu\, w_t}_{0}$$

$\partial_t w$

$$= \frac{1}{2}\frac{d}{dt}\int_U (w_t^2 + |\nabla w|^2)\,dx$$

$$E(t) = \int_U \underbrace{w_t^2(x,t)}_{\text{kinetic}} + \underbrace{|\nabla w(x,t)|^2}_{\text{potential}}\,dx$$

$0 = \frac{1}{2}\frac{d}{dt}E(t)$, i.e. $E(t) \equiv \text{const}$

$\forall t \in (0,T]$, $E(t) = E(0) = 0$

$\Rightarrow \forall t \in (0,T]$, $w_t(x,t) \equiv 0$

$x \in U$, $|\nabla w(x,t)| \equiv 0$

$\Rightarrow w \equiv \text{const}$ on U_T.

by boundary cond. $\Rightarrow w \equiv 0$

Rmk / Cor of Proof. If w solves $\begin{cases} (\partial_{tt} - \Delta)w = 0 & \text{on } U_T \\ w = 0 & \text{on } \partial U \times [0,T] \\ w = g,\ w_t = h & \text{on } U \times \{t=0\} \end{cases}$

make sure $w_t = 0$ after IBP

then $E(t) = E(0)$

$$= \int_U h^2 + |\nabla g|^2 \neq 0.$$

Conservation of energy

(Now say $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C^1(\mathbb{R}^n \times [0,\infty))$

solves $(\partial_{tt} - \Delta)u = 0$

$E(t) = \int_U u_t^2(x,t) + |\nabla u(x,t)|^2\,dx$. fix $U \subset \mathbb{R}^n$ open bounded

$$\frac{d}{dt}E(t) \cdot \frac{1}{2} = \int_U (u_t \cdot u_{tt} + \nabla u \cdot \nabla u_t)\,dx$$

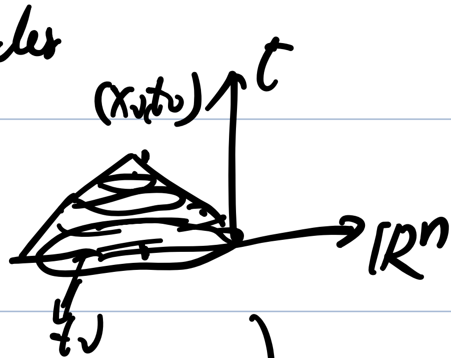
$$= \int_U u_t \underbrace{(u_{tt} - \Delta u)}_{=0}\,dx + \int_{\partial U} u_t \cdot \nabla u \cdot \nu$$

check?

$$\leq \int_{\partial U} \left(\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right)$$ Cauchy-Schwarz.

Fix $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$, let's consider $B(x_0, t_0 - t)$ → radius $= t_0 - t$

and let $e(t) = \int_{B(x_0, t_0-t)} u_t^2 + |\nabla u|^2 \, dx$



$$\frac{1}{2}\frac{d}{dt} e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} \frac{d}{dt}(u_t^2 + |\nabla u|^2)\,dx + \frac{1}{2}\int_{\partial B(x_0,t_0-t)} (u_t^2 + |\nabla u|^2)(-\nu\cdot\nu)$$

moving domain. $e(t)$

IBP

$$= \int_{\partial B(x_0,t_0-t)} u_t \nabla u\cdot\nu - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |\nabla u|^2$$

$$\leq \int_{\partial B(x_0,t_0-t)} \frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{2} \ldots$$

$$= 0$$

ex: what is right energy $(\partial_{tt} - c^2\Delta)u = 0$

So $\forall t \leq t_0$, $e(t) \leq e(0)$

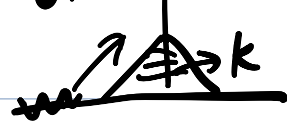
Cor (Stability)

$$\int_{B(x_0,t_0-t)} |\partial_t u(x,t)|^2 + |\nabla u(x,t)|^2 \, dx \leq \int_{B(x_0,t_0)} \partial_t u(x,0)^2 + |\nabla u(x,0)|^2 \, dx$$

because decreasing of $e(t)$

Cor 2. Finite speed of propagation.

If $u = u_t = 0$ in $B(x_0,t_0) \times \{t=0\}$, then $u \equiv 0$ in $K = \{(x,t) : x \in B(x_0, t_0 - t)\}$



Lec 29

wave eqn in 3D

$(\partial_{tt} - \partial_{xx}) w = (\partial_t - \partial_x)(\partial_t + \partial_x) w \to$ in 1D

Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ solve

$$\begin{cases} (\partial_{tt} - \Delta) u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g, \; u_t = h & \text{on } \mathbb{R}^n \times (0,\infty) \end{cases}$$

For $n \geq 2$, $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$

Let $U(x; r, t) := \fint_{\partial B(x,r)} u(y,t)\, dS(y) \to *$

frozen

$G(x; r) := \fint_{\partial B(x,r)} g(y)\, dy$

$H(x; r) = \fint_{\partial B(x,r)} h(y)\, dy$

Lemma Let $x \in \mathbb{R}^n$ be fixed. Then U solves

$$\begin{cases} \partial_{tt} U - \partial_{rr} U - \frac{n-1}{r} \partial_r U = 0 & \text{on } \mathbb{R}_+ \times (0,\infty) \\ U = G, \; U_t = H & \text{on } \mathbb{R}_+ \times \{t = 0\} \end{cases}$$

Proof Exercise - similar to proof of MVP for harmonic fns //.

Euler-Poisson-Darboux eqn.

Thm (Kirchoff's Formula) Let $n=3$ and
let $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solve.

$$\begin{cases} \partial_{tt} u - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty) \\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{t=0\} \end{cases}$$

Then. u is given by

$$u(x,t) = \fint_{\partial B(x,t)} \left[t h(y) + g(y) + \nabla g(y) \cdot (y-x) \right] dS(y)$$

Rmk: Loss of regularity. $g \in C^k$.
u is only $\in C^{k-1}$
because of ∇g

$\int_{\partial B} \to$ not FCT.

Proof - Let $\underline{\hat{U}(r) = U \cdot r}$. Note $\hat{U}(0) = 0$

Note $\frac{\partial}{\partial r} \hat{U}(0) = \lim_{r \to 0} \frac{\hat{U}(r)}{r} = \lim_{r \to 0} U(r) \xrightarrow{*}$ $r \to 0$
$= u(x,t)$

Claim $\hat{U}$ solves wave equ.

$$\hat{U}_{rr} = \frac{d}{dr}[U_r \cdot r + U]$$

$$= U_{rr} \cdot r + 2U_r$$

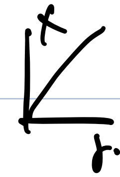
$$\hat{U}_{tt} = r U_{tt}$$

$$\hat{U}_{tt} - \hat{U}_{rr} = r\left\{ U_{tt} - U_{rr} - 2\frac{U_r}{r} \right]$$

if $n=3$.
$2 = n-1$

$= 0 \to$ lemma. (EPD).

So $\hat{u}$ solves
$$\begin{cases} (\partial_{tt} - \partial_{rr}) \hat{U} = 0 & \text{on } \mathbb{R}_+ \times (0,\infty) \\ \hat{U} = \hat{G}, \ \hat{U}_t = \hat{H} & \text{on } \mathbb{R}_+ \times \{t=0\} \\ \hat{U} = 0 & \text{on } \{r=0\} \times (0,\infty) \end{cases}$$
$\to \lim_{r\to 0} \hat{U}(x;r) = 0$

For $t>r$, sol'n formula tells us
$$\hat{U}(x;r,t) = \frac{1}{2}\left[\hat{G}(t+r) - \hat{G}(t-r) + \int_{t-r}^{t+r} \hat{H}(y)\,dy\right]$$


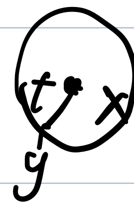
$$u(x,t) = \lim_{r\to 0} \frac{\hat{U}(x;r,t)}{r} \qquad + \hat{G}(t) - \hat{G}(t)$$
$$= \frac{1}{2}\left\{\lim_{r\to 0} \frac{\hat{G}(t+r) - \hat{G}(t-r)}{r} + \lim_{r\to 0} \frac{1}{r}\int_{t-r}^{t+r} \hat{H}(y)\,dy\right]$$
$$= \underbrace{\hat{G}'(t)}_{\to G(x;t).} + \hat{H}(t)$$
$$= \frac{d}{dt}\left[t ⨍_{\partial B(x,t)} g(y)\,dS(y)\right] + t ⨍_{\partial B(x,t)} h(y)\,dS(y)$$
$\frac{d}{dt} = ?$
$$= ⨍_{\partial B(x,t)} g(y)\,dS(y) + t ⨍_{\partial B(x,t)} \nabla g(y) \frac{(y-x)}{|y-x|}\,dS(y) + ⨍_{\partial B(x,t)} t\,h(y)\,dS(y)$$
$$= ⨍_{\partial B(x,t)} \left[g(y) + \nabla g(y)(y-x) + t\,h(y)\right] dS(y) \;//.$$

Pf. of ① $\frac{d}{dt}$ ②

$$\frac{d}{dt}⨍ g(y)\,dS(y) = \frac{d}{dt}\frac{1}{n\alpha_n t^{n-1}}\int_{\partial B(x,t)} g(y)\,dS(y)$$

$$= \frac{d}{dt}\frac{t^{n-1}}{n\alpha_n t^{n-1}}\int_{\partial B(0,1)} g(x+tz)\,\underline{dS(z)} \qquad t^{n-1}$$

$z = \frac{y-x}{t}$



$y = x + zt$

$S(y) = t^n S(z) + C.$

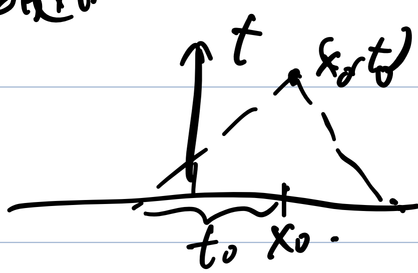
$dS(y) = n t^{n-1} dS(z)$

$$= \frac{1}{n\alpha_n}\int_{\partial B(0,1)} \nabla g(x+tz)\cdot z\cdot dS(z)$$

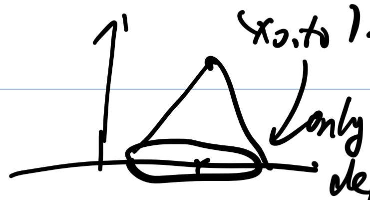
$$= \frac{1}{n\alpha_n t^{n-1}}\int_{\partial B(x,t)} \nabla g(y)\,\frac{y-x}{t}\,dS(y) \quad \text{change back}.$$

$dz = \frac{1}{n t^{n-1}}\,dy.$

$$= ⨍_{\partial B(x,t)} \underline{\qquad\qquad}.$$



1D.



only depends on $\partial B(x_0, t_0)$ (sphere surface in 3D)

3D

2D: $u(x,y,z) = u(x,y)$

t = constant

but not trivial because $u(x,y)$ depends on 3D sphere.

Lec 30

Let $u \in C^2(\mathbb{R}^2 \times [0,\infty))$ solve

$$\begin{cases} (\partial_{tt}-\Delta)u = 0 & \text{on } \mathbb{R}^2\times(0,\infty) \\ u=g,\ u_t=h & \text{on } \mathbb{R}^2\times\{t=0\}\end{cases}$$

Add extra variable, $\bar u(x_1,x_2,x_3,t) = u(x_1,x_2,t)$

$$\bar u \in C^2(\mathbb{R}^3\times[0,\infty))$$

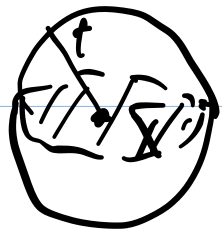
$\bar g(x,y,z) = g(x,y)$. $\bar h(x,y,z) = h(x,y)$

$\bar u$ solves $\begin{cases} (\partial_{tt}-\Delta)\bar u = 0 & \text{in } \mathbb{R}^3\times(0,\infty) \\ \bar u = \bar g \quad \bar u_t = \bar h & \text{on } \mathbb{R}^3\times\{t=0\}\end{cases}$

$\Rightarrow$ Let $x=(x_1,x_2)\in\mathbb{R}^2$. $\bar x = (x_1,x_2,0)$

Kirchoff $u(x,t) = \bar u(\bar x, t)$

$$= \frac{d}{dt}\left(t ⨍_{\partial \bar B(\bar x,t)} \bar g\, d\bar S\right) + t ⨍_{\partial\bar B(\bar x,t)} \bar h\, d\bar S$$



$$\Phi(y) = \Big(y, \underbrace{\sqrt{t^2-|x-y|^2}}_{f(y)}\Big)$$

$$J\Phi(y) = \sqrt{1+|\nabla f(y)|^2}$$

$\nabla f(y) = \dfrac{-(x-y)}{\sqrt{t^2-|x-y|^2}}$. $|\nabla f|^2+1 = 1 + \dfrac{|x-y|^2}{t^2-|x-y|^2}$

$= t^2$

$$\cdots = \frac{\cdots}{t^2-|x-y|^2} = \frac{1}{\sqrt{1-\left(\frac{|x-y|}{t}\right)^2}}$$

$$t \fint_{\partial \bar{B}(\bar{x},t)} \bar{h}\, d\bar{S} = \frac{t}{4\pi t^2}\int_{\partial B(\bar{x},t)} \bar{h}\, d\bar{S} = \frac{2}{4\pi t}\int_{B(x,t)} h(y)\frac{1}{\sqrt{1-\left(\frac{|x-y|}{t}\right)^2}}\, dy$$

$$= \frac{1}{2\pi t}\int_{B(x,t)} \frac{t\, h(y)}{\sqrt{t^2-|x-y|^2}}$$

$$= \frac{1}{2}\fint_{B(x,t)} \frac{t^2 h(y)}{\sqrt{t^2-|x-y|^2}}$$

$$\frac{\partial}{\partial t}\left(t \fint_{\partial \bar{B}(x,t)} \bar{g}\, d\bar{S}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi t}\int_{B(x,t)} g(y)\frac{1}{\sqrt{1-\left(\frac{|x-y|}{t}\right)^2}}\, dy\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi t}\int_{B(0,1)} g(x+tz)\frac{1}{\sqrt{1-|z|^2}}\, dz\right) \quad \left(z = \frac{y-x}{t}\right)$$

$$y = x+tz$$

$$= \frac{1}{2\pi}\int_{B(0,1)} g(x+tz)\frac{1}{\sqrt{1-|z|^2}}\, dz + \frac{t}{2\pi}\int_{B(0,1)} \frac{\nabla g(x+tz)\cdot z}{\sqrt{1-|z|^2}}$$

$$= \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{g(y)}{\sqrt{1-\frac{|x-y|^2}{t^2}}}\, dz + \frac{t}{2\pi}\int_{B(x,t)} \frac{\nabla g(y)\cdot\left(\frac{y-x}{t}\right)}{\sqrt{1-\frac{|x-y|^2}{t}}}\, dy$$

$$= \frac{1}{2}\fint_{B(x,t)} \frac{t g(y) + t\nabla g(y)\cdot(y-x)}{\sqrt{t^2-|x-y|^2}}\, dy$$

$$u(x,t) = \frac{1}{2}\fint_{B(x,t)} \frac{t g(y) + t\nabla g(y)\cdot(y-x) + t^2 h(y)}{\sqrt{t^2-|x-y|^2}}\, dy$$

Poisson's formula

Hadamard's method of descent

Rmk. in 3d Kirchoff's formular. → odd dim (surphere) integral

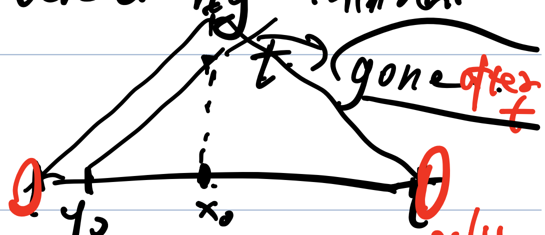
$$u(x,t) = \fint_{\partial B(x,t)} \cdots$$

in 2d. Poisson's formular. → even dim (solid ball) integral.

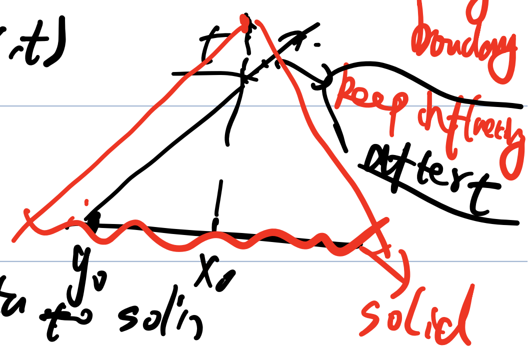
$$u(x,t) = \fint_{B(x,t)} \cdots$$

Huygen's Principle.

In 3D, sol'n is only inflenced by initial data on $\partial B(x,t)$



In 2D, sol'n is inflenced by initial data in $B(x,t)$



Rmk In 2D, we also have loss of regularity from initial data to sol'n

$$u(x,t) = \fint \nabla g$$

Lec 30.5

Inhomogeneous wave equation - Duhamel Formula

$$\begin{cases}(\partial_{tt}-\Delta)u = f & \text{on } \mathbb{R}^n\times(0,\infty)\\ u=0,\ u_t=0 & \text{on } \mathbb{R}^n\times\{t=0\}\end{cases}$$

$u=g,\ u_t=h$

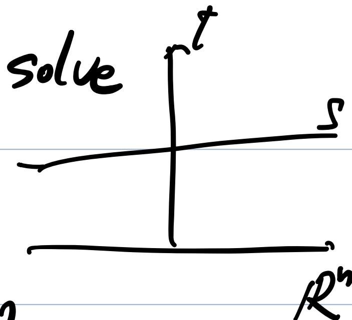
sol'n for $u_1 + \underbrace{u_2}$ $\quad (\partial_{tt}-\Delta)u=0$

sol'n for.

(linearity) $\{\cdots = f$

sol'n to $\} = g = h$

For each $s>0$, let $\tilde{u}(x,t;s)$ solve



$$(\Delta)\begin{cases}(\partial_{tt}-\Delta)\tilde{u}=0 & \text{on } \mathbb{R}^n\times(s,\infty)\\ \tilde{u}(x,t=s;s)=0 \ \cancel{f(x,s)} & \text{on } \mathbb{R}^n\times\{t=s\}\\ \tilde{u}_t(x,t=s;s)=f(x,s)\ \cancel{0} & \text{on } \mathbb{R}^n\times\{t=s\}\end{cases}$$

Define. $\quad u(x,t) := \int_0^t \tilde{u}(x,t;s)\,ds \quad (*)$

Thm Let $f\in C^2(\mathbb{R}^n\times[0,\infty))$ $n=1,2,3$

[C^1 enough in $n=1$]

Then $u\in C^2(\mathbb{R}^n\times[0,\infty))$ and solves

in $(*)$

$(\partial_{tt}-\Delta)u=f$ on $\mathbb{R}^n\times(0,\infty)$ with $u(x,t)\to 0$, $u_t(x,t)\to 0$ as $(x,t)\to(x_0,0)$

Proof. D'Alembert / Poisson / Kirchhoff
$n=1$ $n=2$ $n=3$

$\Rightarrow \tilde{u}(\cdot;s) \in C^2(\mathbb{R}^n \times [s,\infty))$ (with $f \in C^2$ ...)

$\Rightarrow u = C^2(\mathbb{R}^n \times [0,\infty))$

$\cdot\ \partial_t u(x,t) = \underbrace{\tilde{u}(x,t;t)}_{=0 \text{ by } (\Delta) \text{ def.}} + \int_0^t \partial_t \tilde{u}(x,t;s)\,ds$

take derivative w.r.t. t in (*)

$\Rightarrow$ initial conditions hold in limit sense

$\partial_{tt} u(x,t) = \underbrace{\partial_t (x,t;t)}_{=f(x,t)} + \int_0^t \partial_{tt} \tilde{u}(x,t;s)\,ds$

$\Delta u(x,t) = \int_0^t \Delta \tilde{u}(x,t;s)\,ds$

def in space "x" not "t"

$(\partial_{tt} - \Delta) u = f(x,t) + \int_0^t \underbrace{(\partial_{tt} - \Delta)\tilde{u}(x,t;s)}_{=0}\,ds$
$= f(x,t)$

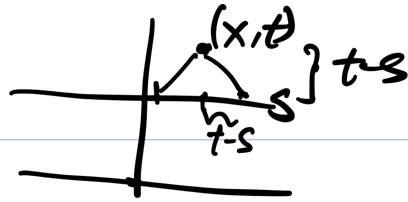
A more useful form when $n=1$. ($\tilde{u}$ is easy to find)

$u(x,t) = \int_0^t \tilde{u}(x,t;s)\,ds$ where $\tilde{u}(x,t;s)$ solves

$$\begin{cases} (\partial_{tt} - \partial_{xx})\tilde{u}(x,t;s) = 0 & \text{on } \mathbb{R} \times (s,\infty) \\ \tilde{u} = 0,\ \tilde{u}_t = f & \text{on } \mathbb{R} \times \{t=s\} \end{cases}$$

$$\Rightarrow \tilde{u}(x,t;s) = \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy$$

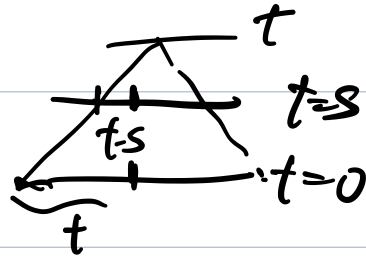
(D'Alembert)



$$\Rightarrow u(x,t) = \int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds$$

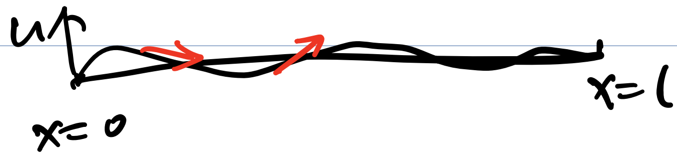
$$= \iint_K f(y,s)\,dy\,ds$$

where $K = \{(y,s) : |x-y| < t-s\}$



Derivation in one space dim.

Take an elastic string, ends clamped to peg distance l apart and assume string is flexible no bending energy



Let $u(x,t)$ denote vertical displacement of point x at time t, Assume motion is entirely transverse

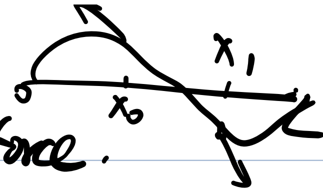
The force is tension.

Tension acts tangentially along the string with magnitude $T(x,t)$

$$\tau(x,t) = \frac{(1, u_x)}{\sqrt{1+u_x^2}} \to \text{vector}: (F_{all} = T\cdot\tau)$$

Vertical component of the force is $\dfrac{T\cdot u_x}{\sqrt{1+u_x^2}}$

Newton's 2^{nd} Law $F=ma$.



Take $x_0 x_1$ Total transverse force.

$u+$. positive.

$$\frac{T(x_1,t)\,u_x(x_1,t)}{\sqrt{1+u_x(x_1,t)^2}} - \frac{T(x_0,t)\,u_x(x_0,t)}{\sqrt{1+u_x(x_0,t)^2}} = \int_{x_0}^{x_1} \underbrace{\rho}_{\text{density.}}\, u_{tt}\, dx$$

FTC.

$$\int_{x_0}^{x_1} \partial_x \left(\frac{T(x,t)\cdot u_x(x,t)}{\sqrt{1+u_x(x,t)^2}} \right)$$

$$\Updownarrow$$

$$\rho\, u_{tt}(x,t).$$

Assume. motion is small, i.e. u_x small

$\underline{u_x(x,t)} \ll 1$ Denominator ≈ 1

Assume tension magnitude is constant T_0.

$\Rightarrow \partial_x(T_0 u_x) = T_0 u_{xx} = \rho u_{tt}$

$\Rightarrow u_{tt} = \underbrace{\frac{\rho}{T_0}}_{c^2} \partial_{xx} u$

c^2. c is the speed wave front

$c = \sqrt{\frac{\rho}{T_0}}$