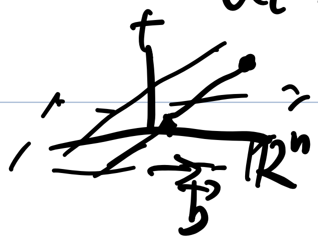
Transport Equation, $b \in \mathbb{R}^n$ fixed

$$\star\ u_t + \nabla u \cdot b = 0 \qquad u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$$



if u solves $\star$

then letting $z(s) = u(x+sb, t+s)$

$$\dot z(s) = \nabla u(x+sb, t+s)\cdot b + \partial_t u(x+sb, t+s) = C$$

So given $b \neq 0 \in \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}$

$$\text{(IVP)} \quad \begin{cases} u_t + b \cdot \nabla u = 0 & \text{in } \mathbb{R}^n \times \{t > 0\} \\ u(x, 0) = g(x) \end{cases}$$

Def: $u(x,t) = g(x - tb)$

if $u \in C^1$, then u solves $\star$

if a C^1 solution $\star$ exists, it must be u

Now, non-homogeneous $u_t + \nabla u \cdot b = f(x, t)$

$$\dot z(s) = \nabla u(x+sb, t+s)\cdot b + \partial_t u(x+sb, t+s)$$
$$= f(x+sb, t+s), \qquad z(s) = u(x+sb, t+s)$$

IVP

$$\star \quad \begin{cases} u_t + b \cdot \nabla u = f(x,t) & \text{on } \mathbb{R}^n \times \{t > 0\} \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

$$u(x,t) = z(0) = z(-t) + \int_{-t}^{0} \dot z(r)\,dr$$
$$= g(x - tb) + \int_{-t}^{0} f(x + rb, t + r)\,dr$$

$$= g(x-tb) + \int_0^t f(x+(s-t)b, s)\, ds$$

$(s = t + r)$.

So $\begin{cases} \text{if } u \text{ is } C^1 \text{ then it solves } \star. \\ \text{if } \exists \text{ a } C^1 \text{ sol'n to } \star, \text{ it must be } u. \end{cases}$

Lec 31.

First Order PDE. (linear one: transport)

Ex. Gas transported along velocity field

$U \subset \mathbb{R}^n$ open bounded

$\rho : U \times [0,T] \to \mathbb{R}$ density of gas at $x \in U$ time t

$v : U \times [0,T] \to \mathbb{R}^n$ velocity of gas particle at x time t

Fix $V \subset U$ smooth boundary. Mass of gas in V

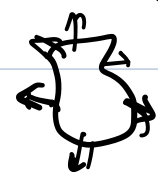
$$m(t,V) = \int_V \rho(x,t)\,dx$$

$$\int_V \partial_t \rho(x,t)\,dt = \frac{d}{dt} m(t,V) = -\int_{\partial V} \rho(x,t)\, v(x,t) \cdot \nu_V(x)$$

ν_V: outer unit normal

divergence thm.

$$= -\int_V \operatorname{div}(\rho(x,t) v(x,t))\,dx$$



$$\partial_t \rho(x,t) + \operatorname{div}(\rho(x,t) v(x,t)) = 0$$

continuity equ.

if v is divergence free (gas is incompressible)

$$\Rightarrow \partial_t \rho(x,t) + \nabla \rho(x,t) \cdot v(x,t) = 0$$

If we know the velocity field $v(x,t)$

Can we construct the solution $\rho(x,t)$ given initial cond?

Yes. Let $X(t)$ solve

$$\begin{cases} \frac{d}{dt} X(t) = v(X(t),t) \\ X(0) = x_0 \quad (\text{fix}) \end{cases}$$

$$\frac{d}{dt}\rho(X(t),t) = \underset{\nabla_x \rho \cdot \dot{X}}{\nabla\rho \cdot v} + \partial_t \rho = 0 \quad \text{if } \rho \text{ solves continuity eqn.}$$

i.e. ρ is constant along particle path.

$$\rho(X(t),t) = \rho(X(0),0)$$

(transport eqn $v = c$).

$F : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ given

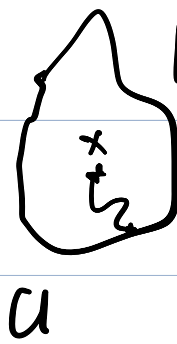
$$F(\nabla u(x), u(x), x) = 0 \quad \text{in } U \subset \mathbb{R}^n \text{ open.}$$

$$u = g \quad \text{on } \Gamma \subset \partial U$$

Ex. Eikonal eqn. $|\nabla u| - 1 = 0$

$$F(p,z,x) = |p| - 1$$

Method of characteristics



U

Let $x(s) = (x_1(s) \dots x_n(s))$ be curve $\dot{x}(s) = \text{TBD}$

Let u solve $\begin{cases} F(\nabla u, u, x) = 0 \text{ on } U \end{cases}$

$$z(s) = u(x(s)) \in \mathbb{R}$$

$$p(s) = \nabla u(x(s)) = (p_1(s) \dots p_n(s)) \in \mathbb{R}^n$$

$$\dot{z}(s) = \nabla u(x(s)) \cdot \dot{x}(s) = p(s) \cdot \dot{x}(s) \quad ②$$

$$\dot{p}_i(s) = \frac{d}{ds} u_{x_i}(x(s)) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \boxed{\dot{x}_j(s)}$$

$$\dot{p}(s) = D^2 u(x(s)) [\dot{x}(s)]$$ → not good. never closed. ∴ more derivative $p(s)$

$$D_p F = (F_{p_1}, F_{p_2} \cdots F_{p_n}) . \; D_z F = F_z , \; D_x F = (F_{x_1}, \cdots F_{x_n})$$

$$0 = \frac{\partial}{\partial x_i} F(\nabla u(x), u(x), x)$$

$$= \sum_{j=1}^{n} F_{p_j} u_{x_j x_i} + F_z u_{x_i} + F_{x_i}$$

$$\Rightarrow \sum_{j=1}^{n} F_{p_j} u_{x_i x_j} = -F_z u_{x_i} - F_{x_i}$$

Set $\dot{x}_j(s) = F_{p_j}(p(s), z(s), x(s))$ ①

$$\dot{x}(s) = D_p F(p(s), z(s), x(s))$$

$$\dot{p}_i(s) = \sum_{j=1}^{n} u_{x_i x_j}(x(s)) \underbrace{\dot{x}_j(s)}_{\Leftrightarrow F_{p_j}} = -F_z u_{x_i} - F_{x_i}$$

$$\dot{p}(s) = -D_z F \, p(s) - D_x F \quad ③$$

So ① ② ③ is a system of ODE in $\mathbb{R}^{2n+1}$

$$
(*)\begin{cases}
\dot{x}(s) = D_p F(p(s), z(s), x(s)) \\
\dot{z}(s) = p(s)\cdot \dot{x}(s) \\
\dot{p}(s) = -D_z F(p(s), z(s), x(s))\, p(s) - D_x F(p(s), z(s), x(s))
\end{cases}
$$

Thm. Assume $u \in C^2(U)$ solves starting points.

$$F(\nabla u, u, x) = 0 \text{ in } U$$

If $x(s)$ solves $\dot{x}(s) = D_p F(\nabla u(x(s)), u(x(s)), x(s))$

Then $z(s) = u(x(s))$ and $p(s) = \nabla u(x(s))$

solves the characteristic equs. $(*)$.

Lec 32

$$\dot{x}(s) = D_p F(p(s), z(s), x(s))$$

$$\dot{z}(s) = p(s)\cdot D_p F(p(s), z(s), x(s))$$

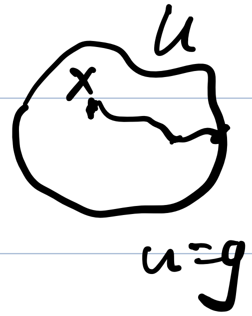
$$\dot{p}(s) = -D_z F(p(s), z(s), x(s))\,p(s) - D_x F(p(s), z(s), x(s))$$

Characteristic equations

$$\begin{cases} F(\nabla u(x), u(x), x) = 0 \text{ in } U \\ u(x) = g(x) \text{ on } \Gamma \subset \partial U \end{cases} \quad \in C^1$$

$$p(s) = \nabla u(x(s))$$

$$z(s) = u(x(s))$$



Conservation Laws.

$$u_t(x,t) + f(u(x,t))_x = 0 \quad x \in \mathbb{R},\ t>0$$

Ex. Inviscid Burger's Eqn. $f(s) = \frac{1}{2}s^2$ → compressible ~~fluid~~ gas -velocity

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = u_t + u\cdot u_x = 0$$

IVP $\begin{cases} u_t + f(u)_x = 0 & \text{on } \mathbb{R}\times\{t>0\} \\ u(x,0) = u_0(x) & \text{for } x\in\mathbb{R} \end{cases}$

Anytime u is a C^1 sol'n to IVP we can write eqn as

$$u_t + \underbrace{f'(u)}_{a(u)}\cdot u_x = 0$$

$$F(\partial_x u, \partial_t u, u, x, t) = 0 \text{ for } F(\overset{u_x}{\cancel{p_x}}, \overset{u_t}{\cancel{p_t}}, \overset{u}{z}, x, t)$$

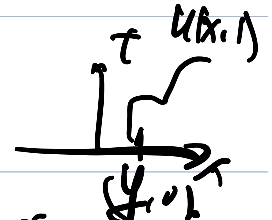
$$F(\overset{u_x}{p_x}, \overset{u_t}{p_t}, \overset{u}{z}, x, t) = \underbrace{(\overset{u_x}{p_x}\ \ \overset{u_t}{p_t})}_{p}(a(z), 1)$$

$$D_pF = (a(z), 1)$$

$$(\dot{x}(s), \dot{t}(s)) = (a(z(s)), 1)$$

$$\dot{x}(s) = a(z(s)), \quad \dot{t}(s) = 1$$

↳ choose $t(s) = s$



For any fixed $y \in \mathbb{R}$.

solving ODE for x w/ $x(0) = y$

$$x(t) = y + \int_0^t a(z(s))\,ds$$

$$\begin{cases} \dot{z}(s) = p(s)\cdot(a(z(s)), 1) \\ p(s) = (\partial_x u(x(s)), \partial_t u(x(s))) \end{cases}$$

u solves eqn. $F = 0$

$= 0$

For F in this form, let's solve charactic eqns.

Curve: solve

$$(\dot{x}(s), \dot{t}(s)) = (a(z(x(s))), 1)$$

with $(x(0), t(0)) = (y, 0)$

$$\dot{z}(s) = (p_x(s), p_t(s))\cdot(z(x(s)), 1)$$

$= 0$ since u solves eqn.

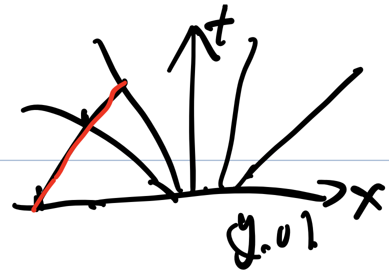
z constant

$$z(s) = z(0) = u_0(y)$$

$$\begin{cases} \dot{x}(s) = a(z(s)) = a(u_0(y)), \\ \dot{x}(0) = y \end{cases}$$

$$\underline{x(s) = y + s\,a(u_0(y))}$$

$$\begin{cases} z(t) = u_0(y) \\ x(t) = y + t\, a(\underbrace{u_0(y)}_{\text{depend on } y}) \end{cases}$$



Prop. Let $f \in C^1$, $a = f'$, consider consev. law

$$(*) \begin{cases} u_t + a(u)\, u_x = 0 & \text{on } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{on } \mathbb{R} \end{cases}$$

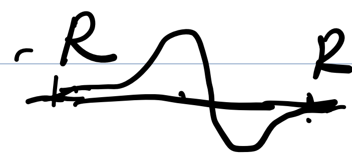
Assume $u_0 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then $(*)$ has a C^1 solution u def for all time

no collide.

$\Leftrightarrow$ the map $y \mapsto a(u_0(y))$ is nondecreasing

Ex Burgers's $u_t + \frac{1}{2}(u^2)_x = 0$

$$u_t + u\, u_x = 0$$



$$a(u_0) = u_0$$

If $u_0(y)$ is constant for $|y| > R$ but not globally constant.

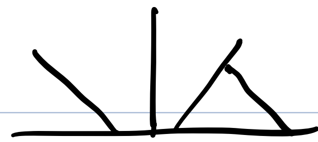
no global C^1 sol'n to IVP exists

Lec 33

Conservation Laws

$$(*)\begin{cases}\partial_t u + f(u)_x = 0 & x\in\mathbb{R},\ t>0\\ u(x,0)=u_0(x) & x\in\mathbb{R}\end{cases}$$

A C^1 solution to $(*)$ exist only for very special initial data ($f\in C^1$)



$\Leftrightarrow f'(u_0(x))$ non-decreasing (no collision)

Def We say u is a weak sol'n to $(*)$ if $\forall \varphi\in C^1(\mathbb{R}\times[0,\infty))$, we have

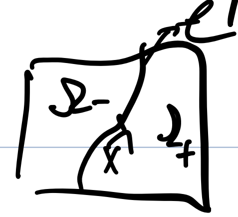
$$0=\int_0^\infty\int_{\mathbb{R}}\varphi_t u+\varphi_x f(u)\,dx\,du+\int_{\mathbb{R}}\varphi(x,0)u_0(x)\,dx$$

Ex Let $u\in C^1(\mathbb{R}\times[0,\infty))$ solve $(*) \Rightarrow$ u is a weak sol'n to $(*)$

Set up

Hypothesis (H$*$) Suppose in a rectangle $\Omega=(x_1,x_2)\times(t_1,t_2)$ there is a C^1 curve $t\mapsto\hat{x}(t)$ s.t. $u\in C^1(\Omega_-)$, $u\in C^1(\Omega_+)$ and u is uniformly continuous on Ω_+ and Ω_-.

$$\Omega_- := \{(x,t)\in\Omega : x<\hat{x}(t)\}$$
$$\Omega_+ := \{(x,t)\in\Omega : x>\hat{x}(t)\}$$



$e=\mathrm{im}\begin{pmatrix}1\\ \hat{x}'\end{pmatrix}$

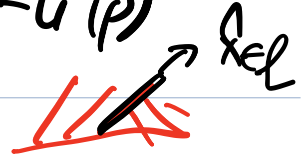
Note (H✠) $\Rightarrow \forall (x_0,t_0)\in \mathcal{C}$ $\qquad \mathcal{C}=\{(\hat{x}(t),t) \mid t\in(t_1,t_2)\}$

$u^{\pm}(x_0,t_0) := \lim\limits_{(x,t)\in\Omega_{\pm}} u(x,t)$ exists

and $t\mapsto u^{\pm}(\hat{x}(t),t)$ is continuous

Notation at $p\in\mathcal{C}$, let $[u](p)=u^+(p)-u^-(p)$

$$\sigma(p)=\frac{d\hat{x}}{dt}(t_0)$$



Thm. Assume H✠.

Then u is a weak sol'n to $u_t+f(u)_x=0$ in Ω

$\Leftrightarrow$ (1) $u_t+f(u)_x=0$ in $\Omega_-\cup\Omega_+$

(2) $-\sigma[u]+[f(u)]=0$ on $\mathcal{C}$

$$-\sigma(u^+-u^-)+f(u^+)-f(u^-)=0$$

φ is a test fn in Ω

test fns φ compacted support in Ω

Rmks condition (2) is called the Rankine–Hugoniot condition.

As long as $u^+-u^-\neq 0$ (2) reads

$$\sigma=\hat{x}'(t)=\frac{f(u^+)-f(u^-)}{u^+-u^-}$$

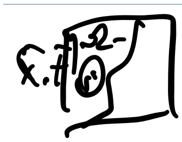
Pf. $\Rightarrow$ Sps u sat (H✠) and is a weak sol'n in Ω

Step 1. Fix $(\bar{x},\bar{t})\in\Omega_-$ (WLOG)

wts $u_t+f(u)_x=0$ at $(\bar{x},\bar{t})$

Assume not, WLOG $[u_t+f(u)_x]\big|_{(\bar{x},\bar{t})}>0$

and since $u\in C^1$ and Ω_- is open, we can find $r>0$



s.t. $B((\bar{x},\bar{t}),r)\subset\Omega_-$ and $u_t+f(u)_x>0$ in this ball. "$\mathbb{B}$"

Take $\varphi \in C_c^1(B)$, $\varphi \geq 0$ and not $\equiv 0$

then $0 = \int_B u\,\partial_t \varphi + f(u)\,\partial_x \varphi \, dx\,dt$

$(f(u),u)\cdot \nabla_{(x,t)}\varphi$

$$\overset{IBP}{=} \int_B \varphi \underbrace{\mathrm{div}_{(x,t)}(f(u),u)}_{\partial_x f(u) + \partial_t u > 0} \, dx\,dt + 0 > 0$$

Step2. Fix $p = (\bar{x}(t), t) \in \mathcal{C}$, choose $\varepsilon > 0$ s.t. $B(p,\varepsilon) \subset \Omega$, choose $\varphi \in C_c^1(B(p,\varepsilon))$

$$0 = \int_{B(p,\varepsilon)} (f(u),u)\cdot \nabla_{(x,t)}\varphi$$

$$= \int_{B_-} (f(u),u)\cdot \nabla_{(x,t)}\varphi + \int_{B_+} (f(u),u)\cdot \nabla_{(x,t)}\varphi$$

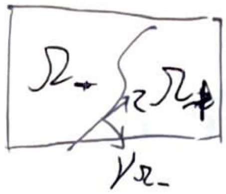
$$\overset{IBP\,(div)}{=} \int_{B_-} \underbrace{\mathrm{div}_{(x,t)}(f(u),u)}_{\substack{= f(u)_x + \partial_t u = 0 \\ \text{by step1}}} \varphi \, dx\,dt + \int_{B\cap \mathcal{C}} \varphi_{(x,t)} (f(u^-),u^-)\cdot \nu_{B_-}$$

$$+ \int_{B_+} \underbrace{\quad\cdots\quad}_{=0} + \int_{B\cap\mathcal{C}} \varphi_{(x,t)} (f(u^+),u^+)\cdot \nu_{B_+}$$

$$= \int_{B\cap\mathcal{C}} \varphi_{(x,t)} \left(f(u^-) - f(u^+),\, u^- - u^+\right)\cdot \nu_{B_-}$$

$$\Rightarrow \left(f(u^-) - f(u^+),\, u^- - u^+\right)\cdot \underbrace{\nu_{B_-}} = 0$$

Lec 34 $\Omega = (x_1, x_2) \times (t_1, t_2)$



$\ell = (\hat{x}(t), t)$ — shock wave (discontinuity).

$t \mapsto \hat{x}(t)$ C^1.

$u \in C^1(\Omega_- \cup \Omega_+)$

unif cts on Ω_-, Ω_+

Thm. u is a weak sol'n to $u_t + f(u)_x = 0$ in Ω

$\Leftrightarrow$ (1) u solves $u_t + f(u)_x = 0$ on $\Omega_- \cup \Omega_+$

(2) $-\sigma[u] + [f(u)] = 0$ on ℓ.

$\sigma = x'(t)$

$[u] = u^+ - u^-$

if $[u] \neq 0$, $\sigma = \dfrac{[f(u)]}{[u]}$

Pf: we have showed (1) and

$(f(u^-) - f(u^+), u^- - u^+) \nu_{\Omega_-} = 0$ on φ

$$\tau = \frac{(\hat{x}'(t), 1)}{\sqrt{1 + |\hat{x}'(t)|^2}} \Rightarrow \nu_{\Omega_-} = \frac{(1, -\hat{x}'(t))}{\sqrt{1 + |\hat{x}'(t)|^2}}$$

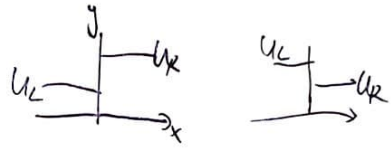
$$\Rightarrow -[f(u)] + \underbrace{\hat{x}'(t)}_{\sigma}[u] = 0$$

$\Leftarrow$ all steps reversible //.

Ex. Assume $f \in C^2$, strictly convex

eg. inviscid Burgers: $f(s) = \frac{1}{2}s^2$

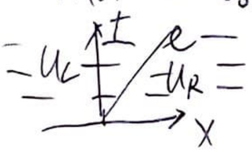
initial data $u_0(x) = \begin{cases} u_L(x), & x < 0 \\ u_R(x), & x > 0 \end{cases}$



Goal: Find weak sol'n to $\begin{cases} u_t + f(u)_x = 0 & \text{on } \mathbb{R} \times \mathbb{R}_+ \\ u(x) = u_0(x) & \text{on } \mathbb{R} \end{cases}$

Let $$\sigma_0 = \frac{f(u_0^+(0)) - f(u_0^-(0))}{u_0^+(0) - u_0^-(0)} = \frac{f(u_R) - f(u_L)}{u_R - u_L}$$

$\hat{x}(t) = 0 + \sigma_0 t$



$u(x,t) = \begin{cases} u_L, & x < \hat{x}(t) \\ u_R, & x > \hat{x}(t) \end{cases}$

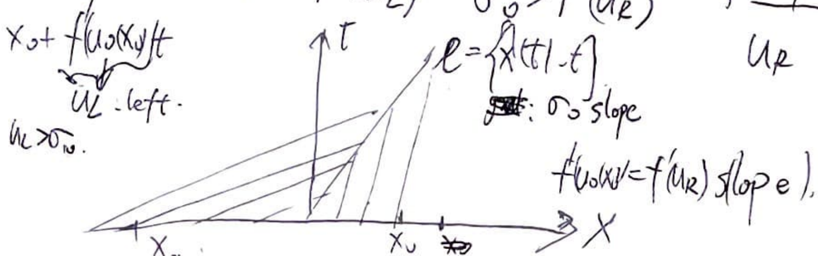
is a weak sol'n to IVP

Rankine-Hugoniot: $\sigma_0 = \dfrac{[f(u)]}{[u]}$

What do characteristics look like?

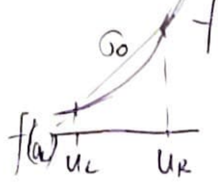
Case 1. $u_L > u_R$

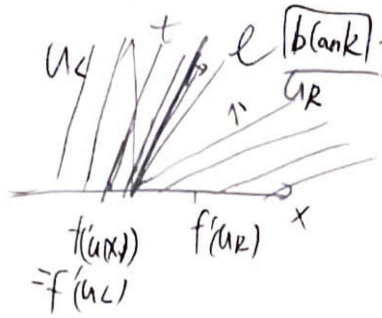
strict convexity. $f'(u_L) > \sigma_0 > f'(u_R)$



Case 2. $u_L < u_R$

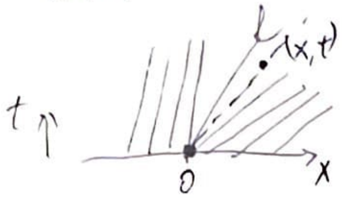
$f(u_L) < \sigma_0 < f(u_R)$





$\boxed{\text{blank}}$ → not depend on inti data but the shock.

Another weak sol'n for case 2



"Rarefaction fan"

$x = 0 + a(u(x,t))t$ $\boxed{a := f'}$.

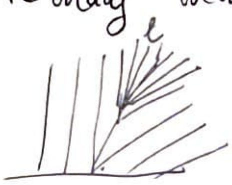
slope: $\frac{x}{t} = a(u)$.

f: strictly convex $\Rightarrow f' = a$ invertible $\Rightarrow u(x,t) = a^{-1}\left(\frac{x}{t}\right)$

Def. $u(x,t) = \begin{cases} u_L & \text{if } x < a(u_L)t \\ a^{-1}\left(\frac{x}{t}\right) & a(u_L)t < x < a(u_R)t \\ u_R & \text{if } x > a(u_R)t \end{cases}$ u const along rays

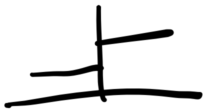
Check by hand: same proof R-H condition $\Rightarrow$ u is a weak sol'n since u is continuous in $\mathbb{R} \times (0,\infty)$ since a^{-1} cts.

infinite many weak ~~sol~~ sol'ns, from any (x,t) on ℓ.



Lec 35

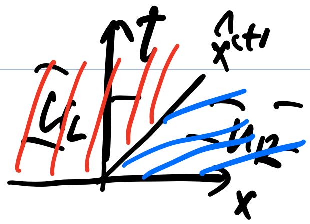
Ex $f \in C^2$, strictly convex



$$u_0 = \begin{cases} u_L & x<0 \\ u_R & x>0 \end{cases}$$

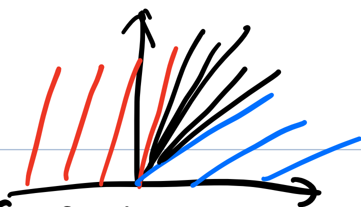
Case 2 $u_L < u_R$

Soln 1



$\hat{x}(t) = \sigma t$, $\sigma = \dfrac{f(u_R) - f(u_L)}{u_R - u_L}$

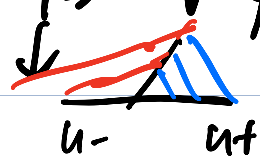
Sol'n 2.



$$u(x,t) = \begin{cases} u_L, & x < a(u_L)t \\ a^{-1}\left(\frac{x}{t}\right), & a(u_L) < \frac{x}{t} < a(u_R) \\ u_R, & x > a(u_R)t \end{cases} \qquad a = f'$$

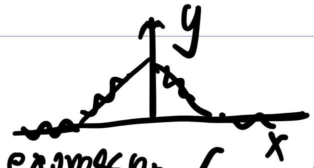
Def A weak sol'n to $u_t + f(u)_x = 0$ that is discontinuous along a C^1 curve is deemed admissible $\mathcal{C} = \{(\hat{x}(t), t) = p\}$ shockcurve and is called a shockwave if

(Lax entropy condition) $f'(u^-(p)) > \sigma > f'(u^+(p))$. $\forall\, p \in \mathcal{C}$



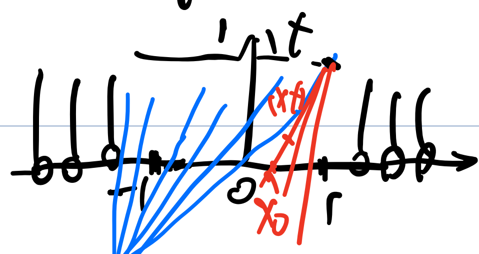
Ex. $f(s)=\frac{1}{2}s^2 \longrightarrow f'(s)=s=a(s),\ a^{-1}(s)=s$

$u_0(x)=\max\{0,1-|x|\} \rightarrow$



Goal: write down explicit expression for admissible weak sol'n to IVP

$$\begin{cases} u_t+u\cdot u_x=0 \\ u(x,0)=u_0(x)\end{cases}$$



$(-1,-1)$

Compression wave

$x(t)=x_0+\underbrace{u_0(x_0)}t$

$$=\begin{cases} 0, & |x|>1 \\ 1-|x| & |x|<1 \end{cases}$$

$$\begin{cases} 1+x_0, & x_0<0 \\ 1-x_0 & x_0>0 \end{cases}$$

$x=x(t)=x_0+(1-x_0)t$

$=x_0(1-t)+t$

$\frac{x-t}{1-t}=x_0$

$u(x,t)=u_0(x_0)$

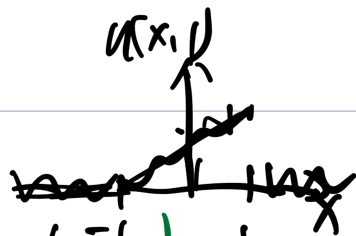
$=1-x_0=\frac{1-x}{1-t}$

for $t\in(0,1)$

$$u(x,t)=\begin{cases} 0, & x<-1,\ x>1 \\ \frac{x-(-1)}{t-(-1)}, & -1<x<t \\ \frac{1-x}{1-t}, & t<x<1 \end{cases}$$

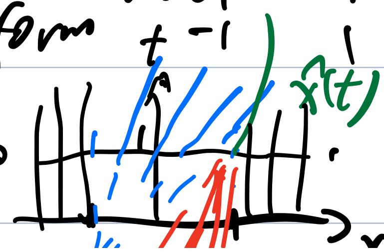
After $t=1$.

$$u(x,1)=\begin{cases} 0 & x<-1,\ x>1 \\ \frac{x+1}{2} & -1<x<1 \end{cases}$$



For $t>1$, seek a solution of form

$$u(x,t)=\begin{cases} 0 & x<-1 \\ \frac{x+1}{t+1}, & -1<x<\hat{x}(t) \\ 0 & x>\hat{x}(t) \end{cases} \rightarrow$$



To make a weak sol'n, need to satisfy $\hat{R-H}$

$$\sigma = \hat{x}'(t) = \frac{[f(u)]}{[u]} = \frac{\frac{1}{2}(u^+)^2 - \frac{1}{2}(u^-)^2}{u^+ - u^-} = \frac{1}{2}u^+ + \frac{1}{2}u^-$$

$$= \frac{1}{2}\left(0 + \frac{\hat{x}+1}{t+1}\right)$$

$\hat{x}(t)$ det by solving

$$\begin{cases} \hat{x}'(t) = \frac{1}{2}\frac{\hat{x}(t)+1}{t+1} \\ \hat{x}(1) = 1 \end{cases} \Rightarrow \hat{x}(t) = \sqrt{2}\sqrt{(t+1)} - 1$$

admissible.

$$0 < \sigma = \frac{1}{2}\frac{x+1}{t+1} < \frac{x+1}{t+1}$$

Lax-Entropy cond ✓.

Lec 36 Assume $F \in C^2$. $\partial U \in C^3$. $g \in C^3$

BVP $\begin{cases} F(Du, u, x) = 0 & \text{in } U \\ u = g & \text{on } \Gamma \subset \partial U \end{cases}$

local existence theory for BVP

$\begin{cases} \dot{x}(s) = F_p(p(s), z(s), x(s)) \\ \dot{z}(s) = p(s) \cdot F_p(p(s), z(s), x(s)) \\ \dot{p}(s) = -F_x(p(s), z(s), x(s)) \end{cases}$

① Locally flatten the boundary.

Fix $x_* \in \Gamma \subset \partial U$

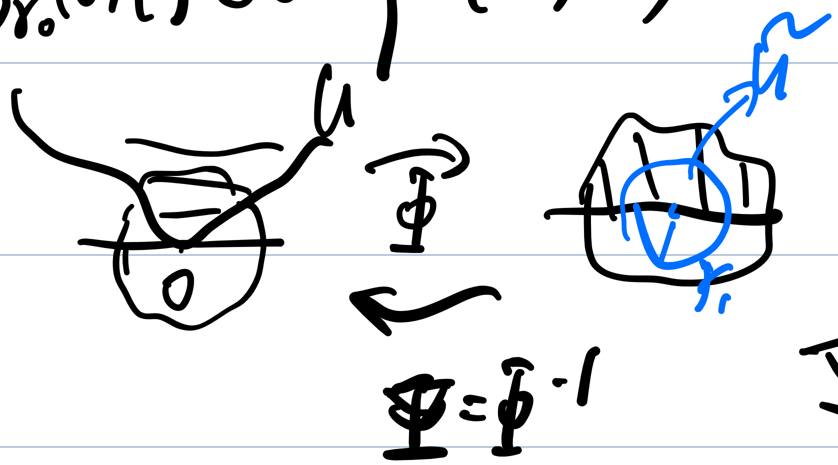
WLOG. assume

$x_* = 0$ and

$\exists\, \gamma : \mathbb{R}^{n-1} \to \mathbb{R}$

$\gamma \in C^3$ $D\gamma(0^{\in \mathbb{R}^{n-1}}) = 0_{\in \mathbb{R}^{n-1}}$ $\exists\, r_0 > 0$. s.t.

$B_{r_0}(0) \cap \partial U = \{ (x', x_n) : x_n > \gamma(x')\ \ x' \in D_{r_0}(0) \}$



$\Phi(x', x_n) = (x', x_n - \gamma(x'))$

$\Psi(y', y_n) = (y', y_n + \gamma(y'))$

$\Phi\ \Psi \in C^3$

How does eqn transform accordingly?

Suppose u is a C^1 sol'n to BVP

For $y \in \tilde{U}$ let $\tilde{u}(y) = u(\Psi(y))$

i.e. $u(x) = \tilde{u}(\Phi(x))$

$$\frac{\partial}{\partial x_i} u(x) = \sum_{j=1}^{n} \frac{\partial \tilde{u}}{\partial x_j}(\Phi(x)) \frac{\partial \Phi_j}{\partial x_i} \Rightarrow (D\Phi)_{ji}$$

$$\nabla u(x) = \nabla \tilde{u}|_{\Phi(x)} \cdot D\Phi|_x = (D\Phi_x)^T \nabla \tilde{u}|_{\Phi(x)}$$

$$0 = F(\nabla u(x), u(x), x)$$

$$= F(\nabla \tilde{u}|_{\Phi(x)} D\Phi|_x, \tilde{u}(\Phi(x)), x))$$

$y = \Phi(x)$

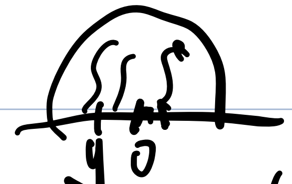
$$= F(\nabla \tilde{u}(y) D\Phi|_{\Psi(y)}, \tilde{u}(y), \Psi(y)))$$

Letting $\tilde{F}(p, z, y) = F(p D\Phi|_{\Psi(y)}, z, \Psi(y))$

Then $\tilde{u}$ solves $\begin{cases} \tilde{F}(\nabla \tilde{u}, \tilde{u}, y) = 0 \text{ in } \tilde{U} \\ \tilde{u}(y) = \tilde{g}(y) = g(\Psi(y)) \text{ on } \tilde{\Gamma} \end{cases}$

on $\tilde{\Gamma}$

$\tilde{F} \in C^2,\ \tilde{g} \in C^3,\ \tilde{\Gamma} \in C^\infty$ upshot: It suffices to prove local existence in this flat setting.

② 

Set up initial data for characteristic eqns.

We hope to solve for $(x(s), z(s), p(s))$ which will give us $(x(s), u(x(s)), \nabla u(x(s)))$

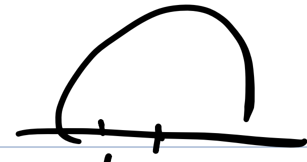
So starting from $y = (\bar{y}, 0) \in \Gamma$, want to set

$x(0) = x^0 = y,\ z(0) = z^0 = g(y),\ p_i(0) = p_i^0 = \frac{\partial g}{\partial x_i}(y)$ $i = 1, \dots, n-1$

In order to solve $F(\nabla u, u, x) = 0$ we need to choose $p_n(0) = p_n^0$ s.t.

$$F(p^0, z^0, x^0) = 0$$

It may be that no such p_n^0 exists, or that infinitely many exist



A triple $(p^0, z^0, x^0) \in \mathbb{R}^{2n+1}$ is called a compatible triple at $y \in \Gamma$ if

$F(p^0, z^0, x^0) = 0,\ x^0 = y,\ z^0 = g(y),\ p_i^0 = \partial_i g(y)$, $i = 1, \dots, n-1$.

Suppose (p^0, z^0, x^0) is a compatible triple at 0.
What conditions can guarantee that we can find compatible triples

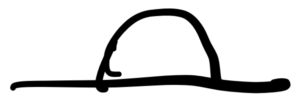
$$(p^0(y), z^0(y), x^0(y)) \text{ for all } y \in \Gamma \text{ in nbhd.}$$

of origin 0?

We want: $x^0(y) = y$, $z^0(y) = g(y)$, $p_i^0(y) = \partial_i g(y)$ for $i = 1, \dots, n-1$

Rephased: What condition guarantees we can define $p_n^0(y)$ st. $F(p^0(y), z^0(y), y) = 0$.

Lemma. Assume $F_{p_n}(p^0, z^0, x^0) \neq 0$ Then.
$\exists!$ sol'n $y \mapsto p_n^0(y)$ for $y \in \Gamma$ in nbhd of 0 (neighborhood)
s.t. $F(p^0(y), z^0(y), y) = 0$



Proof. Define $H(q \in \mathbb{R}, y \in \mathbb{R}^{n-1} \times \{0\})$

$$H(q, y) = F(\partial_1 g(y), \dots, \partial_{n-1} g(y), q, g(y), y)$$

$$H(p_n^0, 0) = 0$$

$$\partial_q H(p_n^0, 0) = F_{p_n}(p^0, z^0, x^0) \neq 0$$
↳ hypothesis

$H \in C^2$. implicit fn. thm $\Rightarrow$
$\forall y \in \Gamma$ in nbhd of 0. $\exists!$ $q(y)$ st. $H(q(y), y) = 0$ //

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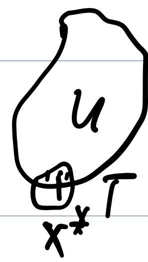
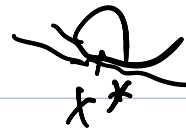
$$\begin{cases} F(\nabla u, u, x) = 0 & \text{in } U \subset \mathbb{R}^n \text{ (open)} \\ u = g & \text{on } \Gamma \subset \partial U \end{cases}$$



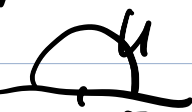
Local existence

Standing Hypotheses. $g, F \in C^3$, $\partial U \in C^4$



$$\dot{x}(s) = F_p(p(s), z(s), x(s))$$
$$\dot{z}(s) = p(s) \cdot F_p(p(s), z(s), x(s))$$
$$\dot{p}(s) = -F_x(p(s), z(s), x(s))$$

Step 1. WLOG.



$\Gamma \subset \mathbb{R}^{n-1} \times \{0\} = \mathbb{R}^{n-1}$

Step 2. what data to feed charact. eqn.

For $y \in \Gamma$, let $x^0(y) = y$, $z^0 = g(y)$

$p_i^0(y) = \partial_i g(y)$ $i = 1, \dots, n$

Assume $p_n^0(0)$ can be chosen st $F(p^0(0), z^0(0), x^0(0)) = 0$

i.e. $(p^0(0), z^0(0), x^0(0))$ is compatible triple

Assume $F_{p_n}(p^0(0), z^0(0), x^0(0)) \neq 0$

$\Rightarrow \exists!$ C^2 fn $y \mapsto p_n^0(y)$ in a neighborhood of 0 in Γ

s.t. $(p^0(y), z^0(y), x^0(y))$ is a compatible triple.

(p^0, z^0, x^0) is non-characteristic.

Note $y \mapsto (p^0(y), z^0(y), x^0(y))$ is C^2

Step3. Solve charater equs.

$X(s) = (x(s), z(s), p(s))$ want to solve.

(*) $\dot{X}(s) = G(X(s))$. $G: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is C^2

$\exists ! \Rightarrow$ for each $y \in \Gamma$, $\exists !$ solution $X(s)$ to (*) with $X(0) = (p^0(y), z^0(y), x^0(y))$ on $(T_-(y), T_+(y))$

By stability, up to shrinking Γ, we have

$T_+(y) \geq \frac{1}{2} T_+(0)$

$T_-(y) \leq \frac{1}{2} T_-(0)$

$\forall y \in \Gamma$, i.e. for the flow Φ generated by G

$\Gamma \subset U_{\frac{T_+(0)}{2}} \cap U_{\frac{T_-(0)}{2}}$

we showed Φ is C^1 as fn of (p^0, z^0, x^0)

In fact, since $G \in C^2$, Φ is C^2 as fn of (p^0, z^0, x^0) $\qquad G \in C^1$

$\cdot, s) \longleftrightarrow \tilde{\Phi}(y,s)$

Thus. $\Phi(p^0(y), z^0(y), x^0(y)) \longleftarrow y$ is C^2 $\quad x(y,s)$ is x component of $\tilde{\Phi}(y,s)$

Lemma (Local invertibility) Assume still.

$F_{p_n}(p^0(0), z^0(0), x^0(0)) \neq 0$, $\exists$ open interval I containing 0, a neighborhood W of 0 in $\Gamma \subset \mathbb{R}^{n-1}$ and neighborhood V of 0 in $\mathbb{R}^n$ s.t.

for each $x \in V$, $\exists! (y,s) \in W \times I$

s.t. $x = x(y,s)$

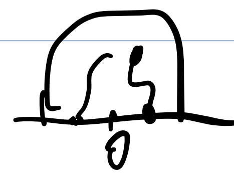
The mapping $x \mapsto (y,s)$ is C^2

Proof. The mapping $(y,s) \mapsto x(y,s)$ is C^2,

$x(0,0) = 0$

if $Dx|_{(0,0)}$ is invertible then lemma follows by Inverse Function Theorem.

$$Dx(y,s)\Big|_{\substack{y=0\\ s=0}} = \begin{array}{c} \begin{array}{ccc|c} \partial y_1 & & \partial y_{n-1} & \partial s \end{array} \\ \left(\begin{array}{ccc|c} 1 & & 0 & F_{p_1} \\ & \ddots & & \vdots \\ 0 & & 1 & F_{p_{n-1}} \\ \hline 0 & 0 & 0 & F_{p_n} \end{array}\right) \end{array}$$



$\det Dx(0,0) = F_{p_n}(p^0(0), z^0(0), x^0(0)) \neq 0$

non-characteristic

Step 4. For $x \in V$

Let $u(x) = z(y(x), s(x))$

Thm. The function u is $C^2(V)$ and solves $F(\nabla u(x), u(x), x) = 0$ for $x \in V$ with $u(x) = g(x)$ on $W \subset \Gamma$

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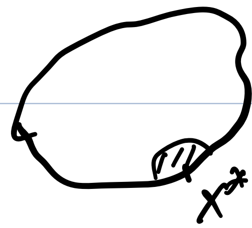
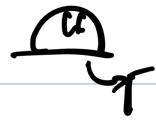
$$\begin{cases} F(\nabla u, u, x) = 0 & \text{in } U \\ u = g & \text{on } \Gamma \subset \partial U \end{cases}$$



Goal: local existence

① WLOG 

②+③ $\exists$ nbhds $\Gamma \subset \mathbb{R}^{n-1}$ of 0 and $(-s_0, s_0)$ of 0 s.t. $\forall y \in \Gamma$, $s \in (-s_0, s_0)$, $\exists$

$\phi(y,s) = p(y,s),\ z(y,s),\ x(y,s)$

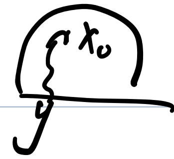
solving char eqns on $(-s_0, s_0)$ with
$$\begin{cases} \dot{x}(s) = F_p \\ \dot{z} = p F_p \\ \dot{p} = -F_x - pF_z \end{cases}$$

$\Phi(y,0) = (p^0(y), z^0(y), x^0(y))$, compatible triple.

Assume. ① $\exists$ compatible triple (p^0, z^0, x^0) at 0

② $F_{p_n}(p^0, z^0, x^0) \neq 0$

$\exists$ nbhd V of 0 in $\mathbb{R}^n$ s.t. $\forall x_0 \in V$,
$\exists! (y,s) \in \Gamma \times I$ s.t. $x(y,s) = x_0$



Set $u(x) = z(y(x), s(x))$

$p(x) = p(y(x), s(x))$

Thm. The fn u is C^2 and solves $F(\nabla u, u, x) = 0$ in $\tilde{V} \subset V$, $u = g$ on $\tilde{\Gamma} \subset \Gamma$

Lemma 1: $f(y,s) = F(p(y,s), z(y,s), x(y,s))$

$\forall s \in I, y \in \tilde{\Gamma} \subset \Gamma$, $f(y,s) \equiv 0$

i.e. $\forall x \in \tilde{\Gamma} \times I$ $\quad F(p(x), u(x), x) = 0$

$\tilde{V} = x(\tilde{\Gamma}, I)$

Lemma 2. $p(x) = \nabla u$

Proof of Lemma 1. compatible triple.

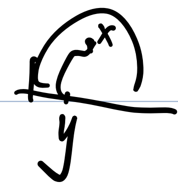
- for $s=0$, $f(y,0) = F(p^0(y), z^0(y), x^0(y)) = 0$
- $\partial_s f(y,s) = F_p \cdot \dot{p}(s) + F_z \cdot \dot{z}(s) + F_x \dot{x}(s)$

$$= F_p \cdot (-F_x - pF_z) + F_z p F_p + F_x F_p$$

$$= 0$$

$f(y,s) \equiv 0$ //

Proof of Lemma 2. $p(x) = p(y(x), s(x))$



$u(x) = z(y(x), s(x))$

$\nabla u(x) = \nabla_y' z \cdot D_x y + \dot{z} \cdot \nabla_x s(x)$

$\dot{z} = p \cdot F_p = p \cdot \dot{x}$ $\quad \nabla_s' z$

Claim: $\nabla_y' z = p \cdot D_y x$

Assuming claim

$$\nabla u(x) = p\left[D_y x\, D_x y + \dot{x}\,\nabla s(x)\right] = p \quad //$$

$\rightsquigarrow x_0 = x(y(x_0), s(x_0))$

$\frac{d}{dx}\Big\{ \; Id = D_y x \cdot D_x y + \dot{x}\,\nabla_x s$

Proof of claim. Fix y. let $r(s) = \nabla_y{}' z(y,s) - p(y,s)\cdot D_y x(y,s)$,

$$r(0) = \nabla_y' g(y) - p^0(y)\cdot\left(\begin{smallmatrix}1 & & \\ & \ddots & \\ & & 1 \\ & 0 & \end{smallmatrix}\right)_{n+1}^{n}$$

$$(\nabla_y' g(y), p_n^0(y))$$

$$= \nabla_y' g(y) - \nabla_y' g(y) = 0$$

$$\dot{r}(s) = \nabla_y' \dot{z} - \dot{p} D_y x - p\cdot D_y \dot{x}$$

$$(\dot{z} = pF_p) \quad = D_y p \cdot \dot{x} - \dot{p} D_y x = D_y p\cdot F_p + (F_x + F_z p)\cdot D_y x$$

$f(y,s) \equiv 0$

$$0 = \nabla_y f(y,s) = F_p D_y p + F_z \nabla_y z + F_x D_y x$$

$$\dot{r}(s) = F_z \underbrace{(p D_y x - \nabla_y' z)}_{-r(s)}$$

$$\Rightarrow \begin{cases} \dot{r} = -F_z \cdot r \\ r(0) = 0 \end{cases}$$

$\Rightarrow r \equiv 0$